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Green's Function of the Linearized Logarithmic Keller–Segel–Fisher/KPP System

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Abstract: We consider a Keller–Segel type chemotaxis model with logarithmic sensitivity and logistic growth. The logarithmic singularity in the system is removed via the inverse Hopf–Cole transformation. We then linearize the system around a constant equilibrium state, and obtain a detailed, pointwise description of the Green's function. The result provides a complete solution picture for the linear problem. It also helps to shed light on small solutions of the nonlinear system.

Keywords: Green's function; fundamental solution; Cauchy problem; Keller–Segel; chemotaxis; logarithmic sensitivity; logistic growth

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1. Introduction

We consider a Keller–Segel type chemotaxis model with logarithmic sensitivity and logistic growth:

$$\begin{cases} c_t = \varepsilon c_{xx} - \mu u c - \sigma c, \\ u_t + \chi[u(\ln c)_x]_x = D u_{xx} + a u(1 - \frac{u}{K}), \end{cases} \qquad x \in \mathbb{R}, \ t > 0. \tag{1}$$

Here, the unknown functions $c = c(x,t)$ and $u = u(x,t)$ are the concentration of a chemical signal and the density of a cellular population, respectively. The system parameters are interpreted as follows.

- $\varepsilon \geq 0$ is the diffusion coefficient of chemical signal.
- $\mu \neq 0$ is the coefficient of density-dependent production/degradation rate of chemical signal.
- $\sigma \geq 0$ is the natural degradation rate of chemical signal.
- $\chi \neq 0$ is the coefficient of chemotactic sensitivity.
- $D \geq 0$ is the diffusion coefficient of cellular population.
- $a \geq 0$ is the natural growth rate of cellular population.
- $K > 0$ is the typical carrying capacity of cellular population.

The system describes the dynamics when certain biological organism releases or consumes a chemical signal in the local environment while both entities are naturally diffusing and reacting. It includes logarithmic chemotactic response of cells to the signal, and some or all of the following mechanisms: random walk/diffusion, consumption/deposition of the chemical by cells, natural degradation of the chemical, and the logistic growth of cells.

Biologically, the sign of χ indicates whether the chemotactic movement is attractive ($\chi > 0$) or repulsive ($\chi < 0$). When $\chi > 0$ and $\mu > 0$, Equation (1) describes the movement of cells that are attracted to and consume the chemical, say, for nutrition. When $\chi < 0$ and $\mu < 0$, as adopted in [1] for the non-growth model, it describes the movement of cells that deposit a chemical signal to modify the local environment for succeeding passages. Such a scenario has found applications in cancer research [2]. Since there is no difference in the analysis of these two scenarios, we assume $\chi\mu > 0$ throughout this paper. Mathematically, the non-diffusive part of the transformed system to be discussed below is hyperbolic in biologically relevant regimes when $\chi\mu > 0$, while it may change type when $\chi\mu < 0$ [3].

The logarithmic singularity in Equation (1) accounts for Fechner's law, which states that subjective sensation is proportional to the logarithm of the stimulus intensity [4]. It can be removed via the inverse Hopf–Cole transformation [5]:

$$v = (\ln c)_x = \frac{c_x}{c}. \tag{2}$$

Under the variables v and u, Equation (1) is converted into

$$\begin{cases} v_t + (\mu u - \varepsilon v^2)_x = \varepsilon v_{xx}, \\ u_t + \chi(uv)_x = Du_{xx} + au(1 - \frac{u}{K}). \end{cases} \tag{3}$$

Equation (3) can be further simplified by rescaling and/or non-dimensionalization:

$$\tilde{t} = \chi\mu K t, \qquad \tilde{x} = \sqrt{\chi\mu K}x, \qquad \tilde{v} = \text{sign}(\chi)\sqrt{\frac{\chi}{\mu K}}v, \qquad \tilde{u} = \frac{u}{K}. \tag{4}$$

After dropping the tilde accent, we arrive at

$$\begin{cases} v_t + (u - \varepsilon v^2/\chi)_x = \varepsilon v_{xx}, \\ u_t + (uv)_x = Du_{xx} + ru(1-u), \end{cases} \tag{5}$$

where

$$r = \frac{a}{\chi\mu K} \geq 0. \tag{6}$$

We consider the Cauchy problem of Equation (1):

$$(c, u)(x, 0) = (c_0, u_0)(x), \tag{7}$$

or equivalently, the Cauchy problem of Equation (5):

$$(v, u)(x, 0) = (v_0, u_0)(x), \tag{8}$$

where the Cauchy datum (v_0, u_0) is assumed to be a small perturbation of a constant equilibrium state $(\bar{v}, \bar{u})$. To be an equilibrium state, we need $\bar{u} = 0$ or $\bar{u} = 1$. It is clear that the former is unstable. Therefore, we set $\bar{u} = 1$. To discuss $\bar{v}$, we apply Equation (2) to have

$$v_0 = \frac{c_0'}{c_0}, \qquad c_0(x) = c_0(0)e^{\int_0^x v_0(y)\,dy} \text{ with } c_0(0) > 0, \tag{9}$$

where for simplicity we have omitted the scaling constant $\text{sign}(\chi)\sqrt{\chi/\mu K}$ from Equation (4). If $v_0 - \bar{v} \in L^1(\mathbb{R})$ while $\bar{v} \gtrless 0$, we have

$$\int_0^\infty v_0(y)\,dy = \pm\infty, \quad \int_{-\infty}^0 v_0(y)\,dy = \pm\infty.$$

Therefore, from Equation (9) we have $c_0(x) \to \infty$ either as $x \to \infty$ or as $x \to -\infty$, depending on $\bar{v} > 0$ or $\bar{v} < 0$. For physically interesting problems, we consider $\lim_{x\to\pm\infty} c_0(x) = c_\pm$ with $0 < c_\pm < \infty$.

Therefore, we take $\bar{v} = 0$. In summary,

$$\lim_{x \to \pm\infty} (v_0, u_0) = (\bar{v}, \bar{u}) = (0, 1). \tag{10}$$

Cauchy problem of Equations (5) and (8) has unique global-in-time small data solution, i.e., when (v_0, u_0) is a small perturbation of $(0, 1)$, see [6,7]. To study small data solutions, especially their long time behavior, one needs to study the corresponding linear system, linearized around the constant equilibrium state. For this, we introduce new variables for the perturbation:

$$w_1 = v, \qquad w_2 = u - 1. \tag{11}$$

Linearizing Equation (5) around $(0, 1)$, we have

$$\begin{cases} w_{1t} + w_{2x} = \varepsilon w_{1xx}, \\ w_{2t} + w_{1x} = D w_{2xx} - r w_2, \end{cases} \tag{12}$$

where $\varepsilon, D, r \geq 0$ are constant parameters.

The goal of this paper is to obtain an accurate and detailed pointwise description, both in x and in t, of the Green's function of Equation (12). The Green's function provides a complete solution picture to Equation (12) and is significant in the linear theory. As discussed above, it also sheds light on the behavior of small data solutions for Equations (5) and (8), which will be studied in a future work.

2. Main Results and Discussion

To obtain the Green's function, we write Equation (12) in vector form:

$$w_t + A w_x = B w_{xx} + L w, \tag{13}$$

where

$$w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}, \quad A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} \varepsilon & 0 \\ 0 & D \end{pmatrix}, \quad L = \begin{pmatrix} 0 & 0 \\ 0 & -r \end{pmatrix}. \tag{14}$$

Here, $\varepsilon, D, r \geq 0$ are constants. We assume that at least one of them is positive. Otherwise, Equation (13) has no dissipation, and its Green's function consists of δ-functions along the characteristic lines, a different scenario to what we discuss below.

The Green's Function of Equation (13) is the solution matrix $G(x, t)$ of

$$\begin{gathered} G_t + A G_x = B G_{xx} + L G \\ G(x, 0) = \delta(x) I_{2\times 2}, \end{gathered} \tag{15}$$

where $\delta(x)$ is the Dirac δ-function, and $I_{2\times 2}$ is the 2×2 identity matrix. Our main results on G are the following theorems, concerning three different cases: $r = 0$; $r > 0$ while $\varepsilon = D = 0$; and $r > 0$ while at least one of ε and D is positive. The cases correspond to different types of systems: hyperbolic–parabolic conservation laws, hyperbolic balance laws, and hyperbolic–parabolic balance laws.

2.1. Hyperbolic–Parabolic Conservation Laws

Theorem 1. *Let $r = 0$, $\varepsilon, D \geq 0$, and at least one of ε and D be positive. Let $l \geq 0$ be an integer. Then, for $x \in \mathbb{R}$, $t > 0$, the Green's function $G(x, t)$ of Equation* (13) *has the following estimates:*

1. *When* $\varepsilon, D > 0$,

$$\frac{\partial^l}{\partial x^l}G(x,t) = \frac{\partial^l}{\partial x^l}\left[\frac{1}{\sqrt{2\pi(\varepsilon+D)t}}e^{-\frac{(x+t)^2}{2(\varepsilon+D)t}}\begin{pmatrix}\frac{1}{2} & -\frac{1}{2}\\ -\frac{1}{2} & \frac{1}{2}\end{pmatrix} + \frac{1}{\sqrt{2\pi(\varepsilon+D)t}}e^{-\frac{(x-t)^2}{2(\varepsilon+D)t}}\begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{pmatrix}\right] + O(1)(t+1)^{-\frac{1}{2}}t^{-\frac{l+1}{2}}e^{-\frac{(x+t)^2}{Ct}} + O(1)(t+1)^{-\frac{1}{2}}t^{-\frac{l+1}{2}}e^{-\frac{(x-t)^2}{Ct}}, \tag{16}$$

   where $C > 0$ *is a constant.*

2. *When* $\varepsilon = 0$ *and* $D > 0$,

$$\begin{aligned}\frac{\partial^l}{\partial x^l}G(x,t) = &\frac{\partial^l}{\partial x^l}\left[\frac{1}{\sqrt{2\pi Dt}}e^{-\frac{(x+t)^2}{2Dt}}\begin{pmatrix}\frac{1}{2} & -\frac{1}{2}\\ -\frac{1}{2} & \frac{1}{2}\end{pmatrix} + \frac{1}{\sqrt{2\pi Dt}}e^{-\frac{(x-t)^2}{2Dt}}\begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{pmatrix}\right]\\ &+ O(1)(t+1)^{-\frac{1}{2}}t^{-\frac{l+1}{2}}e^{-\frac{(x+t)^2}{Ct}} + O(1)(t+1)^{-\frac{1}{2}}t^{-\frac{l+1}{2}}e^{-\frac{(x-t)^2}{Ct}}\\ &+ e^{-t/D}\sum_{j=0}^{l}\delta^{(l-j)}(x)Q_j,\end{aligned} \tag{17}$$

   where $C > 0$ *is a constant, and* Q_j, $0 \le j \le l$, *is a* 2×2, *symmetric, polynomial matrix in* t *with a degree not more than* j. *In particular,*

$$Q_0 = \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}.$$

3. *When* $\varepsilon > 0$ *and* $D = 0$,

$$\begin{aligned}\frac{\partial^l}{\partial x^l}G(x,t) = &\frac{\partial^l}{\partial x^l}\left[\frac{1}{\sqrt{2\pi\varepsilon t}}e^{-\frac{(x+t)^2}{2\varepsilon t}}\begin{pmatrix}\frac{1}{2} & -\frac{1}{2}\\ -\frac{1}{2} & \frac{1}{2}\end{pmatrix} + \frac{1}{\sqrt{2\pi\varepsilon t}}e^{-\frac{(x-t)^2}{2\varepsilon t}}\begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{pmatrix}\right]\\ &+ O(1)(t+1)^{-\frac{1}{2}}t^{-\frac{l+1}{2}}e^{-\frac{(x+t)^2}{Ct}} + O(1)(t+1)^{-\frac{1}{2}}t^{-\frac{l+1}{2}}e^{-\frac{(x-t)^2}{Ct}}\\ &+ e^{-t/\varepsilon}\sum_{j=0}^{l}\delta^{(l-j)}(x)Q_j,\end{aligned} \tag{18}$$

   where $C > 0$ *is a constant, and* Q_j, $0 \le j \le l$, *is a* 2×2, *symmetric, polynomial matrix in* t *with a degree not more than* j. *In particular,*

$$Q_0 = \begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}.$$

Under the assumption $r = 0$, Equation (13) becomes

$$w_t + Aw_x = Bw_{xx}. \tag{19}$$

Green's function estimates on a general system in the form of Equation (19) are detailed in [8] (see Theorems 6.2 and 6.15 therein). It is straightforward to verify that the assumptions of those theorems are satisfied when A and B are given in Equation (14). Therefore, by direct calculation and straightforward application of those theorems, we obtain Theorem 1. We note that Equations (16)–(18) are precise and explicit in the leading terms (and in the singular terms if $l = 0$). We also note that G is symmetric since A and B are, so are Q_j in Equations (17) and (18).

2.2. Hyperbolic Balance Laws

Theorem 2. *Let $r > 0$, $\varepsilon = D = 0$, and $l \geq 0$ be an integer. Then, for $x \in \mathbb{R}$, $t > 0$, the Green's function $G(x,t)$ of Equation (13) has the following estimate:*

$$\begin{aligned}\frac{\partial^l}{\partial x^l}G(x,t) =& \frac{\partial^l}{\partial x^l}\left[\frac{1}{\sqrt{4\pi t/r}}e^{-\frac{rx^2}{4t}}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\right] + O(1)(t+1)^{-\frac{1}{2}}t^{-\frac{l+1}{2}}e^{-\frac{x^2}{Ct}}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\\ &+ (t+1)^{-1}t^{-\frac{l+1}{2}}e^{-\frac{x^2}{Ct}}\begin{pmatrix}O(1) & 0\\ 0 & O(1)\end{pmatrix} + e^{-\frac{r}{2}t}\sum_{j=0}^{l}\delta^{(l-j)}(x+t)Q_{1j}\\ &+ e^{-\frac{r}{2}t}\sum_{j=0}^{l}\delta^{(l-j)}(x-t)Q_{2j},\end{aligned} \tag{20}$$

where $C > 0$ is a constant, and Q_{1j} and Q_{2j}, $0 \leq j \leq l$, are 2×2, symmetric, polynomial matrices in t whose degrees are not more than j. In particular,

$$Q_{10} = \begin{pmatrix}\frac{1}{2} & -\frac{1}{2}\\ -\frac{1}{2} & \frac{1}{2}\end{pmatrix}, \qquad Q_{20} = \begin{pmatrix}\frac{1}{2} & \frac{1}{2}\\ \frac{1}{2} & \frac{1}{2}\end{pmatrix}.$$

Under the assumptions of Theorem 2, Equation (13) becomes

$$w_t + Aw_x = Lw. \tag{21}$$

Green's function estimates on a general system in the form of Equation (21) are detailed in [9] (see Theorem 3.6 therein). It is straightforward to verify that the assumptions of that theorem are satisfied when A and L are given in Equation (14). Therefore, direct application of that theorem would gives us an estimate similar to Equation (20). Here, our result (Equation (20)) has slightly more details in the higher order terms, the second and third terms on the righthand side of Equation (20). This is due to the special structure of A and L in Equation (14), and is justified in Section 3.

2.3. Hyperbolic–Parabolic Balance Laws

Theorem 3. *Let $r > 0$, $\varepsilon, D \geq 0$, and at least one of ε and D be positive. Let $l > 0$ be an integer. Then, for $x \in \mathbb{R}$, $t > 0$, the Green's function $G(x,t)$ of Equation (13) has the following estimates:*

1. *When $\varepsilon, D > 0$,*

$$\begin{aligned}\frac{\partial^l}{\partial x^l}G(x,t) =& \frac{\partial^l}{\partial x^l}\left[\frac{1}{\sqrt{4\pi(\varepsilon+1/r)t}}e^{-\frac{x^2}{4(\varepsilon+1/r)t}}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\right] + O(1)(t+1)^{-\frac{1}{2}}t^{-\frac{l+1}{2}}e^{-\frac{x^2}{Ct}}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\\ &+ (t+1)^{-1}t^{-\frac{l+1}{2}}e^{-\frac{x^2}{Ct}}\begin{pmatrix}O(1) & 0\\ 0 & O(1)\end{pmatrix},\end{aligned} \tag{22}$$

where $C > 0$ is a constant.

2. *When $\varepsilon = 0$ and $D > 0$,*

$$\begin{aligned}\frac{\partial^l}{\partial x^l}G(x,t) =& \frac{\partial^l}{\partial x^l}\left[\frac{1}{\sqrt{4\pi t/r}}e^{-\frac{rx^2}{4t}}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\right] + O(1)(t+1)^{-\frac{1}{2}}t^{-\frac{l+1}{2}}e^{-\frac{x^2}{Ct}}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\\ &+ (t+1)^{-1}t^{-\frac{l+1}{2}}e^{-\frac{x^2}{Ct}}\begin{pmatrix}O(1) & 0\\ 0 & O(1)\end{pmatrix} + e^{-\frac{t}{D}}\sum_{j=0}^{l}\delta^{(l-j)}(x)Q_j,\end{aligned} \tag{23}$$

where $C > 0$ is a constant, and Q_j, $0 \le j \le l$, is a 2×2, symmetric, polynomial matrix in t with a degree not more than $j/2$. In particular,

$$Q_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

3. *When $\varepsilon > 0$ and $D = 0$,*

$$\begin{aligned}\frac{\partial^l}{\partial x^l} G(x,t) =& \frac{\partial^l}{\partial x^l}\left[\frac{1}{\sqrt{4\pi(\varepsilon + 1/r)t}} e^{-\frac{x^2}{4(\varepsilon+1/r)t}} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right] + O(1)(t+1)^{-\frac{1}{2}} t^{-\frac{l+1}{2}} e^{-\frac{x^2}{Ct}} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &+ (t+1)^{-1} t^{-\frac{l+1}{2}} e^{-\frac{x^2}{Ct}} \begin{pmatrix} O(1) & 0 \\ 0 & O(1) \end{pmatrix} + e^{-(r+\frac{1}{\varepsilon})t} \sum_{j=0}^{l} \delta^{(l-j)}(x) Q_j,\end{aligned} \tag{24}$$

where $C > 0$ is a constant, and Q_j, $0 \le j \le l$, is a 2×2, symmetric, polynomial matrix in t with a degree not more than $j/2$. In particular,

$$Q_0 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Comparing Theorems 1–3 we observe that the solution behavior for $r > 0$ is very different to that for $r = 0$. When $r = 0$, from Theorem 1, we see that the leading term in time decay is two heat kernels along the characteristics of A, while, for $r > 0$, from Theorems 2 and 3, it is a heat kernel along t-axis. Therefore, the logistic growth of cells completely changes the solution picture.

From all three theorems, we also observe that the regularity of solution depends solely on the number of nonzero diffusion coefficients ε and D. If both are positive, there is no δ-functions in the Green's function (see Theorems 1 and 3, Case 1). If one of them is zero, then there is a δ-function (and its derivatives as appropriate) (see Theorems 1 and 3, Cases 2 and 3). If both are zero, then there are two δ-functions (see Theorem 2).

The last comment is on the role of D. If there is no logistic growth of cells, the two diffusion coefficients ε and D play the same role (see Theorem 1). However, if there is logistic growth, $r > 0$, then only r and $\varepsilon > 0$ but not D appear in the leading heat kernel (see Theorem 3). That is, logistic growth of cells overwhelms their diffusion.

In next section, we prove Theorem 3 and justify Theorem 2 to finish this paper.

3. Green's Function Estimates

Notation 1. *Throughout this paper, C denotes a universal positive constant, whose value may vary line by line according to the context.*

To study a linear system, we perform Fourier transform with respect to x:

$$\begin{aligned}\hat{w}(\xi,t) &= \int_{\mathbb{R}} w(x,t) e^{-ix\xi}\,dx, \\ w(x,t) &= \frac{1}{2\pi}\int_{\mathbb{R}} \hat{w}(\xi,t) e^{ix\xi}\,d\xi.\end{aligned} \tag{25}$$

Taking Fourier transform of Equation (15), we have

$$\hat{G}_t = E\hat{G}, \qquad \hat{G}(\xi,0) = I_{2\times 2}, \tag{26}$$

where

$$E = E(i\xi) = L - i\xi A - \xi^2 B. \tag{27}$$

Solving Equation (26) gives us

$$\hat{G}(\xi,t) = e^{tE(i\xi)}. \tag{28}$$

Applying the inverse transform, we arrive at

$$\frac{\partial^l}{\partial x^l}G(x,t) = \frac{1}{2\pi}\int_{\mathbb{R}} (i\xi)^l \hat{G}(\xi,t)e^{ix\xi}\,d\xi \tag{29}$$

for an integer $l \geq 0$. Our goal in this section is to estimate the righthand side of Equation (29) to obtain the results in Theorems 2 and 3 under the assumption $r > 0$.

3.1. Spectral Analysis

We carry out spectral analysis of $E(i\xi)$ defined in Equation (27). By straightforward calculation, the eigenvalues of $E(i\xi)$ are

$$\lambda_{1,2}(i\xi) = -\frac{1}{2}[r+(\varepsilon+D)\xi^2] \pm \sqrt{\frac{1}{4}[r+(\varepsilon+D)\xi^2]^2 - \xi^2[(\varepsilon r+1)+D\varepsilon\xi^2]}, \tag{30}$$

and the corresponding eigenprojections are

$$P_{1,2}(i\xi) = \frac{1}{-\xi^2+(\lambda_{1,2}+\varepsilon\xi^2)^2}\begin{pmatrix} -\xi^2 & -i\xi(\lambda_{1,2}+\varepsilon\xi^2) \\ -i\xi(\lambda_{1,2}+\varepsilon\xi^2) & (\lambda_{1,2}+\varepsilon\xi^2)^2 \end{pmatrix}. \tag{31}$$

Therefore, with Equation (28), we have

$$\hat{G}(\xi,t) = e^{tE(i\xi)} = e^{\lambda_1(i\xi)t}P_1(i\xi) + e^{\lambda_2(i\xi)t}P_2(i\xi), \tag{32}$$

with $\lambda_{1,2}$ and $P_{1,2}$ given in Equations (30) and (31).

The leading term in $e^{tE(i\xi)}$ comes from small ξ. Thus, we consider Taylor expansions for $|\xi| \ll 1$:

$$\begin{aligned} &\lambda_1(i\xi) = -(\varepsilon+\frac{1}{r})\xi^2 + O(\xi^4), \qquad \lambda_2(i\xi) = -r + O(\xi^2), \\ &P_1(i\xi) = \begin{pmatrix} 1+O(\xi^2) & -\frac{i\xi}{r}+O(\xi^3) \\ -\frac{i\xi}{r}+O(\xi^3) & O(\xi^2) \end{pmatrix}, \qquad P_2(i\xi) = \begin{pmatrix} O(\xi^2) & O(\xi) \\ O(\xi) & 1+O(\xi^2) \end{pmatrix}. \end{aligned} \tag{33}$$

Similarly, the regularity of G and its derivatives comes from the expansions as $|\xi| \to \infty$. To simplify our formulation, λ_1 takes the positive square root in Equation (30) if $\varepsilon > D$, and the negative one if $\varepsilon < D$, while λ_2 is the other one. For $\varepsilon \neq D$, we have

$$\begin{aligned} &\lambda_1(i\xi) = -D\xi^2 + \lambda_{10}(i\xi), \quad \lambda_2(i\xi) = -\varepsilon\xi^2 + \lambda_{20}(i\xi), \\ &P_1(i\xi) = \begin{pmatrix} p_1 & \frac{p_2}{\varepsilon-D}(i\xi)^{-1} \\ \frac{p_2}{\varepsilon-D}(i\xi)^{-1} & 1-p_1 \end{pmatrix}, \quad P_2(i\xi) = \begin{pmatrix} 1-p_1 & -\frac{p_2}{\varepsilon-D}(i\xi)^{-1} \\ -\frac{p_2}{\varepsilon-D}(i\xi)^{-1} & p_1 \end{pmatrix}, \end{aligned} \tag{34}$$

where

$$\begin{aligned} &\lambda_{10}(i\xi) = -(r+\frac{1}{\varepsilon-D}) + \sum_{j=1}^{\infty} c_{1j}\xi^{-2j}, \quad \lambda_{20}(i\xi) = \frac{1}{\varepsilon-D} + \sum_{j=1}^{\infty} c_{2j}\xi^{-2j}, \\ &p_1 = p_1(i\xi) = \sum_{j=1}^{\infty} \tilde{c}_{1j}\xi^{-2j}, \quad p_2 = p_2(i\xi) = 1 + \sum_{j=1}^{\infty} \tilde{c}_{2j}\xi^{-2j} \end{aligned} \tag{35}$$

are analytic at ∞, with real coefficients $c_{1j}, c_{2j}, \tilde{c}_{1j}$ and $\tilde{c}_{2j}$ in Equation (35).

If $\varepsilon = D$, on the other hand, we have

$$\begin{aligned} \lambda_{1,2}(i\xi) &= -D\xi^2 \pm i\xi - \frac{1}{2}r \pm \lambda_{-1}(i\xi), \\ P_{1,2}(i\xi) &= \begin{pmatrix} \frac{1}{2} \pm p_3 & \mp\frac{1}{2} \pm p_4 \\ \mp\frac{1}{2} \pm p_4 & \frac{1}{2} \mp p_3 \end{pmatrix}, \end{aligned} \tag{36}$$

where

$$\begin{aligned} \lambda_{-1}(i\xi) &= \frac{r^2}{8}(i\xi)^{-1} + (i\xi)^{-1}\sum_{j=1}^{\infty} c_{3j}\xi^{-2j}, \\ p_3(i\xi) &= \frac{r}{4}(i\xi)^{-1} + \sum_{j=2}^{\infty} \tilde{c}_{3j}(i\xi)^{-j}, \quad p_4(i\xi) = \sum_{j=2}^{\infty} \tilde{c}_{4j}(i\xi)^{-j} \end{aligned} \tag{37}$$

are analytic at ∞, and $c_{3j}, \tilde{c}_{3j}$ and $\tilde{c}_{4j}$ are real coefficients.

3.2. Estimates on Inverse Transform

To estimate Equation (29), we focus on the case $r > 0$, $\varepsilon = 0$ and $D > 0$. All other cases are similar, and are discussed at the end of the section. Our goal is to obtain Equation (23). For this, we apply Equations (34) and (35) to have

$$\begin{aligned} (i\xi)^l e^{\lambda_1(i\xi)t}P_1(i\xi) &= O(1)(i\xi)^l e^{-D\xi^2 t + O(1)t}, \\ (i\xi)^l e^{\lambda_2(i\xi)t}P_2(i\xi) &= e^{-\frac{t}{D}}[\sum_{j=0}^{l}(i\xi)^{l-j}Q_j + (i\xi)^{-1}Q_{l+1} + O(1)(i\xi)^{-2}(1 + t + \cdots + t^{\frac{l+l'}{2}}) \\ &\quad + O(1)(i\xi)^{-2-l'}t^{\frac{l+l'+2}{2}}e^{O(1)\xi^{-2}t}] \end{aligned} \tag{38}$$

as $|\xi| \to \infty$. Here, $l \geq 0$ is an integer, $l' = 1$ if l is odd, and $l' = 0$ if l is even. On the other hand, Q_j is a 2×2 polynomial matrix in t with a degree not more than $j/2$, $0 \leq j \leq l+1$. In particular,

$$Q_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

With the same Q_j in Equation (38), we define

$$R^{(l)}(x,t) = \frac{\partial^l}{\partial x^l}G(x,t) - \frac{\partial^l}{\partial x^l}\left[\frac{1}{\sqrt{4\pi t/r}}e^{-\frac{rx^2}{4t}}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right] - e^{-\frac{t}{D}}\sum_{j=0}^{l}\delta^{(l-j)}(x)Q_j. \tag{39}$$

To obtain Equation (23), we need to prove

$$R^{(l)}(x,t) = O(1)(t+1)^{-\frac{1}{2}}t^{-\frac{l+1}{2}}e^{-\frac{x^2}{Ct}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + (t+1)^{-1}t^{-\frac{l+1}{2}}e^{-\frac{x^2}{Ct}}\begin{pmatrix} O(1) & 0 \\ 0 & O(1) \end{pmatrix} \tag{40}$$

for a constant $C > 0$. Using the inverse Fourier transform in Equation (25), we have

$$\begin{aligned} R^{(l)}(x,t) &= \frac{1}{2\pi}\int_{\mathbb{R}} \hat{R}^{(l)}(\xi,t)e^{ix\xi}\,d\xi, \\ \hat{R}^{(l)}(\xi,t) &= (i\xi)^l\left[\hat{G}(\xi,t) - e^{-\frac{1}{r}\xi^2 t}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - e^{-\frac{t}{D}}\sum_{j=0}^{l}(i\xi)^{-j}Q_j\right]. \end{aligned} \tag{41}$$

Lemma 1. *Let $r > 0$, $\varepsilon = 0$ and $D > 0$. For $x \in \mathbb{R}$ and $t > 0$, we have*

$$|R_{12}^{(l)}(x,t)| = |R_{21}^{(l)}(x,t)| \le C(t+1)^{-\frac{1}{2}} t^{-\frac{l+1}{2}} e^{-\frac{x^2}{Ct}} + Ct^{-\frac{l+1}{2}} e^{-\frac{t}{C}}, \tag{42}$$

$$|R_{11}^{(l)}(x,t)| + |R_{22}^{(l)}(x,t)| \le C(t+1)^{-1} t^{-\frac{l+1}{2}} e^{-\frac{x^2}{Ct}} + Ct^{-\frac{l+1}{2}} e^{-\frac{t}{C}}, \tag{43}$$

where $R_{jk}^{(l)}$ denotes the (j,k) entry of $R^{(l)}$, $1 \le j,k \le 2$.

Proof. Let $n > 0$ be small such that Equation (33) applies for $|\xi| \le 2n$, and $N > 0$ be large such that Equation (38) applies for $|\xi| \ge N$. Denote the (k,k') entries of $\hat{G}$ and Q_j as $\hat{G}_{kk'}$ and $Q_{kk'}^{(j)}$, respectively. We write

$$\begin{aligned}
R_{11}^{(l)}(x,t) &= I_1 + I_2 + I_3,\\
I_1 &= \frac{1}{2\pi}\int_{|\xi|\le n} (i\xi)^l [\hat{G}_{11}(\xi,t) - e^{-\frac{1}{r}\xi^2 t}] e^{ix\xi}\, d\xi,\\
I_2 &= \frac{1}{2\pi}\int_{|\xi|\ge N} (i\xi)^l \left[\hat{G}_{11}(\xi,t) - e^{-\frac{t}{D}} \sum_{j=0}^{l} (i\xi)^{-j} Q_{11}^{(j)}\right] e^{ix\xi}\, d\xi,\\
I_3 &= \frac{1}{2\pi}\Bigg[\int_{n\le|\xi|\le N} (i\xi)^l \hat{G}_{11}(\xi,t) e^{ix\xi}\, d\xi - \int_{|\xi|\ge n} (i\xi)^l e^{-\frac{1}{r}\xi^2 t} e^{ix\xi}\, d\xi\\
&\qquad - \int_{|\xi|\le N} e^{-\frac{t}{D}} \sum_{j=0}^{l} (i\xi)^{l-j} Q_{11}^{(j)} e^{ix\xi}\, d\xi\Bigg],
\end{aligned} \tag{44}$$

where all integrals are over subsets of $\mathbb{R}$.

For I_1, we apply Equations (32) and (33) to have

$$I_1 = \frac{1}{2\pi}\int_{|\xi|\le n} (i\xi)^l [e^{\lambda_1(i\xi)t} P_{11}^1(\xi,t) - e^{-\frac{1}{r}\xi^2 t}] e^{ix\xi}\, d\xi + O(1) e^{-\frac{r}{2}t}, \tag{45}$$

where P_{11}^1 is the $(1,1)$ entry of P_1. From Equations (30) and (31), we note that the integrand in Equation (45) is holomorphic in ξ (as a complex variable) in a neighborhood of the origin. Taking n small, we apply Cauchy integral theorem to replace the domain of integration $[-n,n]$ by a path $\Gamma(\alpha) \equiv \{-n + i\eta \,|\, \eta \text{ is from } 0 \text{ to } \alpha\} \cup \{\xi + i\alpha \,|\, -n \le \xi \le n\} \cup \{n + i\eta \,|\, \eta \text{ is from } \alpha \text{ to } 0\}$. Here, α can be positive or negative, but $|\alpha| \le n$. With Equation (33) we have

$$I_1 = \int_{\Gamma(\alpha)} (i\xi)^l [e^{-\frac{1}{r}\xi^2 t + O(|\xi|^4)t} [O(|\xi|^4)t + O(|\xi|^2)] e^{ix\xi}\, d\xi + O(1) e^{-\frac{r}{2}t}.$$

If $r|x|/t \le n$, we set $\alpha = rx/(2t)$. Integrating over each pieces of $\Gamma(\alpha)$, we have

$$\begin{aligned}
|I_1| &\le C\int_{-n}^{n} e^{-\frac{rx^2}{8t} - \frac{\zeta^2 t}{2r}} (|\zeta| + \frac{|x|}{t})^{l+2}\, d\zeta + Ce^{-\frac{n^2}{2r}t} + Ce^{-\frac{r}{2}t}\\
&\le C(t+1)^{-\frac{l+3}{2}} e^{-\frac{x^2}{Ct}} + Ce^{-\frac{t}{C}}.
\end{aligned} \tag{46}$$

If $r|x|/t > n$, we set $\alpha = \frac{n}{2}\,\mathrm{sign}(x)$. The straightforward calculation yields

$$|I_1| \le Ce^{-\frac{t}{C}}. \tag{47}$$

To estimate I_2, we apply Equation (38) to Equation (32) to have

$$\begin{aligned} I_2 &= O(1)\int_{|\xi|\geq N}|\xi|^l e^{-\frac{D}{2}\xi^2 t}\,d\xi + \frac{1}{2\pi}e^{-\frac{t}{D}}\int_{|\xi|\geq N}(i\xi)^{-1}e^{ix\xi}\,d\xi Q_{11}^{(l+1)} \\ &\quad + O(1)e^{-\frac{t}{2D}}\int_{|\xi|\geq N}\xi^{-2}\,d\xi \\ &= O(1)t^{-\frac{l+1}{2}}e^{-\frac{t}{C}}, \end{aligned} \tag{48}$$

noting the second integral on the right-hand side of Equation (48) is the principal value.

To estimate I_3, we write Equation (44)

$$I_3 = \frac{1}{2\pi}\int_{n\leq|\xi|\leq N}(i\xi)^l \hat{G}_{11}(\xi,t)e^{ix\xi}\,d\xi + O(1)t^{-\frac{l+1}{2}}e^{-\frac{t}{C}}. \tag{49}$$

From Equation (30), we note that, for $\xi \in \mathbb{R}\setminus\{0\}$, the real parts of $\lambda_{1,2}$, $\Re(\lambda_{1,2})$, are negative. As a continuous function on a compact set,

$$\Re(\lambda_{1,2})(i\xi) \leq -\beta, \quad \xi \in [-N,-n]\cup[n,N] \tag{50}$$

for a constant $\beta > 0$.

From the characteristic equation of $E(i\xi)$, it is straightforward to verify

$$-\xi^2 + \lambda_{1,2}^2 = \pm\lambda_{1,2}(\lambda_1 - \lambda_2).$$

Noting $\varepsilon = 0$, $P_{1,2}(i\xi)$ in Equation (31) is analytic in any domain of $\mathbb{C}$ where λ_1 and λ_2 are distinct. From Equation (30), there are $\xi_j \in \mathbb{C}$, $1 \leq j \leq 4$, such that $\lambda_1 = \lambda_2$. If a ξ_j is on $(-N,-n)\cup(n,N)$ for $n > 0$ small and $N > 0$ large, we replace $(\xi_j - \alpha, \xi_j + \alpha)$, with $\alpha > 0$ small, by a semi-circle centered at ξ_j with radius α. In this way, we replace $[-N,-n]\cup[n,N]$ by a union of two paths, denoted as Γ. Noting Equation (50) and the continuity of $\Re(\lambda_{1,2})$, we have

$$\Re(\lambda_{1,2})(i\xi) \leq -\frac{\beta}{2}, \quad |P_{1,2}(i\xi)| \leq C, \quad \xi \in \Gamma, \tag{51}$$

by choosing α small. Since the integrand of Equation (49) is an entire function, applying Cauchy theorem and Equation (32), and substituting Equation (51) into Equation (49), we arrive at

$$I_3 = \frac{1}{2\pi}\int_{\Gamma}(i\xi)^l \hat{G}_{11}(\xi,t)e^{ix\xi}\,d\xi + O(1)t^{-\frac{l+1}{2}}e^{-\frac{t}{C}} = O(1)t^{-\frac{l+1}{2}}e^{-\frac{t}{C}}. \tag{52}$$

Here, we have chosen the small semi-circles in Γ on the upper-half complex plane if $x \geq 0$, and lower half-plane if $x < 0$, so that $\Re(ix\xi) \leq 0$.

Combining Equations (44), (46)–(48) and (52), we obtain the estimate for $R_{11}^{(l)}$ in Equation (43). The estimates for $R_{22}^{(l)}$ and $R_{12}^{(l)}$ are obtained in the same way. In particular, the slower decay rate in $R_{12}^{(l)}$ comes from $-i\xi/r$ in the $(1,2)$ entry of $P_1(i\xi)$ in Equation (33), comparing $O(\xi^2)$ in the $(2,2)$ entry. Finally, Equations (14) and (27)–(29) imply that $G(x,t)$ is symmetric. Therefore, Q_j in Equation (38) hence in Equation (23) and $R^{(l)}$ in Equation (39) are symmetric. This gives us $R_{12}^{(l)} = R_{21}^{(l)}$. □

Lemma 2. *Let $K > 0$ be large. Under the assumptions of Lemma 1, for $\xi \in \mathbb{R}$, $t > 0$, and $|x|/t \geq K$, we have*

$$|R^{(l)}(x,t)| \leq Ct^{-\frac{l+1}{2}}e^{-\frac{x^2}{Ct}}, \tag{53}$$

where $R^{(l)}$ is defined in Equation (39).

Proof. From Equations (41) and (28), we have

$$
\begin{aligned}
R^{(l)}(x,t) &= I_4 - \frac{\partial^l}{\partial x^l}\left[\frac{1}{\sqrt{4\pi t/r}} e^{-\frac{rx^2}{4t}} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right] = I_4 + O(1) t^{-\frac{l+1}{2}} e^{-\frac{x^2}{Ct}}, \\
I_4 &= \frac{1}{2\pi}\int_{\mathbb{R}} (i\xi)^l \left[e^{tE(i\xi)} - e^{-\frac{t}{D}} \sum_{j=0}^{l} (i\xi)^{-j} Q_j\right] e^{ix\xi}\, d\xi.
\end{aligned} \tag{54}
$$

Since the integrand of I_4 is an entire function, by a standard argument, we apply Cauchy theorem to replace the integral path $\mathbb{R}$ by $\Gamma_{x/t} = \{\zeta + i\frac{x}{2Dt} \mid \zeta \in \mathbb{R}\}$. Taking K large and applying Equations (32) and (38) gives us

$$
\begin{aligned}
I_4 &= \frac{1}{2\pi}\int_{\Gamma_{x/t}} (i\xi)^l \left[e^{tE(i\xi)} - e^{-\frac{t}{D}} \sum_{j=0}^{l} (i\xi)^{-j} Q_j\right] e^{ix\xi}\, d\xi \\
&= \frac{1}{2\pi}\int_{\Gamma_{x/t}} \Big\{ O(1)(i\xi)^l e^{-D\xi^2 t + O(1)t} + e^{-\frac{t}{D}}[(i\xi)^{-1} Q_{l+1} + O(1)|\xi|^{-2}(1 + t + \cdots + t^{\frac{l+l'}{2}}) \\
&\qquad + O(1)|\xi|^{-2-l'} t^{\frac{l+l'+2}{2}} e^{O(1)|\xi|^{-2}t}]\Big\} e^{ix\xi}\, d\xi \\
&= O(1)\int_{\mathbb{R}} (|\zeta|^l + |\frac{x}{t}|^l) e^{-D\zeta^2 t - \frac{x^2}{8Dt}}\, d\zeta + \frac{e^{-\frac{t}{D}}}{2\pi}\int_{\mathbb{R}} \frac{e^{ix\zeta}}{i\zeta - \frac{x}{2Dt}}\, d\zeta e^{-\frac{x^2}{2Dt}} Q_{l+1} \\
&\qquad + O(1) e^{-\frac{t}{2D} - \frac{x^2}{2Dt}} \int_{\mathbb{R}} \frac{d\zeta}{\zeta^2 + \frac{K^2}{4D^2}}.
\end{aligned}
$$

Note that

$$
\frac{1}{i\zeta - \frac{x}{2Dt}} = \frac{1}{i\zeta} - \frac{x}{2Dt}\left[\frac{1}{\zeta^2 + (\frac{x}{2Dt})^2} + \frac{1}{i\zeta}\frac{\frac{x}{2Dt}}{\zeta^2 + (\frac{x}{2Dt})^2}\right],
$$

which implies

$$
\int_{\mathbb{R}} \frac{e^{ix\zeta}}{i\zeta - \frac{x}{2Dt}}\, d\zeta = \int_{\mathbb{R}} \frac{e^{ix\zeta}}{i\zeta}\, d\zeta + O(1)\Big(|\frac{x}{t}| + |\frac{x}{t}|^2 |x|\Big).
$$

Therefore,

$$
I_4 = O(1) t^{-\frac{l+1}{2}} e^{-\frac{x^2}{Ct}}. \tag{55}
$$

Substituting Equation (55) into Equation (54) gives us Equation (53). □

Combining Lemmas 1 and 2, and noting that G is symmetric, we arrive at Equation (23). The proof of Equation (24) is parallel. The proof of Equation (22) is simpler since Equations (34) and (36) imply that G and its derivatives contain no δ-functions if $\varepsilon, D > 0$. This settles Theorem 3.

Theorem 2 can be either proved as Theorem 3, or derived from the general framework in [9], noting that G is symmetric, and that those $O(\xi^2)$ on the diagonal of $P_1(i\xi)$ in Equation (33) give an extra $(t+1)^{-\frac{1}{2}}$, comparing to $O(\xi)$ in the general framework.

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