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Metrics for Single-Edged Graphs over a Fixed Set of Vertices

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Abstract: Graphs have powerful representations of all kinds of theoretical or experimental mathematical objects. A technique to measure the distance between graphs has become an important issue. In this article, we show how to define distance functions measuring the distance between graphs with directed edges over a fixed set of named and unnamed vertices, respectively. Furthermore, we show how to implement these distance functions via computational matrix operations.

Keywords: graphs; metrics; vertices; directed edges

MSC: 68R10; 05C05; 90B10; 62H30

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1. Introduction

When investigating the measurement of the distances between two sentences or structures, we feel it is necessary to first form a system to measure the distance for tree or graph structures. This was our initial motivation for developing this paper. This will provide a sound foundation and measurement technique for future applications. For example, if there are two separate English sentences, such as S_1 and S_2, and we would like to measure the distance between these two sentences, we identify S_1 with one graph and S_2 with another graph. Their vocabularies in individual sentences can be associated with vertices in the graph. The distance between the vocabularies can be identified with edges of the graphs. Then, we would have constructed graphs for S_1 and S_2. We would then be able to measure the distance between the two graphs. The main purpose of this article is to put forward an approach to define a metric for graphs on a fixed set of vertices.

Suppose V is a set of fixed vertices and E is a set of directed edges. Then, for each edge (v, w), i.e., an edge from v to w, one can assign a value. Since most of the mathematical models can be formalized or represented via vertices and edges, studying the properties of the distances between any two graphs becomes a vital approach to explore the intrinsic properties of a mathematical structure or a real mathematical object [1,2], even being used on some fuzzy objects [3,4]. Some ingenious metrics for handling these fuzzy objects have been explored in depth [5,6]. In this article, we put forward two metrics for graphs with labelled vertices and unlabelled vertices, respectively. Nonetheless, we only consider the directed edges in this article. As for the indirected edges, one can simply treat them as pairs of two directed edges.

2. Definitions and Claims

We use $\mathbb{R}^+$ to denote all the positive real numbers. For any real number α, we use $|\alpha|$ to denote its absolute value. For any set K, we use $\mathcal{P}(K)$ and $|K|$ to denote the power set and the size of K, respectively. If both H and K are sets, we use $H \to K$ to denote the set of all the functions from H to K. We use $H \triangle K$ to denote $(H - K) \cup (K - H)$ (or $H - K \cup K - H$). We call $G = (V, E, W :$

$E \to \mathbb{R}^+$) or in brevity $G = (V, E, W)$ a generalized graph, in which W is a weight function satisfying following conditions:

1. For each $v \in V[(v, v) \in E$ and $W(v, v) = 0]$;
2. For all $(v_1, v_2) \in E[v_1 \neq v_2 \to W(v_1, v_2) > 0]$.

Definition 1. *Let $GG(V)$ denote the set of all the generalized graphs whose vertex sets are exactly V.*

Let $G_1 = (V, E_1, W_1 : E_1 \to \mathbb{R}^+), G_2 = (V, E_2, W_2 : E_2 \to \mathbb{R}^+), G_3 = (V, E_3, W_3 : E_3 \to \mathbb{R}^+) \in GG(V)$ be arbitrary generalized graphs. For any $G = (V, E, W : E \to \mathbb{R}^+) \in GG(V)$ and any $a \in V$, we use $E(a)$ to denote the set of all the endpoints beginning from a, i.e.,

$$E(a) = \{b \in V : (a, b) \in E\}.$$

Furthermore, define the set of all the assigned values of $E(a)$ as follows:

$$W(a) = \{W(a, b) : b \in E(a)\}.$$

3. Metric for Labelled Graphs

In this section, we assume all the vertices in V are labelled. We show how to define a distance between G_1 and G_2 as follows:

Definition 2. *(distance function: labelled vertices, single directed edge) Define $d_1 : GG(V) \times GG(V) \to \mathbb{R}^+$ by*

$$\begin{aligned} d_1(G_1, G_2) := \sum_{a \in V} [\sum_{c \in E_1(a) - E_2(a)} W_1(a, c) + \sum_{c \in E_2(a) - E_1(a)} W_2(a, c) + \\ \sum_{c \in E_1(a) \cap E_2(a)} |W_1(a, c) - W_2(a, c)|]. \end{aligned} \tag{1}$$

Example 1. *Suppose $V = \{v_1, v_2, v_3\}$ is a fixed set of vertices and graph $G_1 = (V, E_1, W_1)$, and $G_2 = (V, E_2, W_2)$, where their vertices assigned to the edges E_1, E_2 and the values for weights W_1, W_2 are given as follows:*

$$\begin{aligned} &E_1 = \{(v_1, v_2), (v_1, v_3), (v_3, v_1), (v_3, v_2)\}, \\ &E_2 = \{(v_1, v_2), (v_2, v_1), (v_2, v_3), (v_3, v_2), (v_3, v_1)\}, \\ &W_1(v_1, v_2) = 4, W_1(v_1, v_3) = 3, W_1(v_3, v_1) = 1, W_1(v_3, v_2) = 7, \\ &W_2(v_1, v_2) = 3, W_2(v_2, v_1) = 5, W_2(v_2, v_3) = 3, W_2(v_3, v_2) = 4, W_2(v_3, v_1) = 6. \end{aligned}$$

Then, the end vertices originating from v_1 via edges in E_1 could be depicted as $E_1(v_1) = \{v_2, v_3\}$. Others follow:

$$\begin{aligned} &E_2(v_1) = \{v_2\}, E_1(v_2) = \emptyset, E_2(v_2) = \{v_1, v_3\}, \\ &E_1(v_3) = \{v_1, v_2\}, E_2(v_3) = \{v_1, v_2\}. \end{aligned}$$

Henceforth, by Definition 1, one could compute the distance for G_1 *and* G_2 *as follows:* $d_1(G_1, G_2) =$

$$
\begin{aligned}
&\sum_{c\in E_1(v_1)-E_2(v_1)} W_1(v_1,c) + \sum_{c\in E_2(v_1)-E_1(v_1)} W_2(v_1,c) + \sum_{c\in E_1(v_1)\cap E_2(v_1)} |W_1(v_1,c) - W_2(v_1,c)| + \\
&\sum_{c\in E_1(v_2)-E_2(v_2)} W_1(v_2,c) + \sum_{c\in E_2(v_2)-E_1(v_2)} W_2(v_2,c) + \sum_{c\in E_1(v_2)\cap E_2(v_2)} |W_1(v_2,c) - W_2(v_2,c)| + \\
&\sum_{c\in E_1(v_3)-E_2(v_3)} W_1(v_3,c) + \sum_{c\in E_2(v_3)-E_1(v_3)} W_2(v_3,c) + \sum_{c\in E_1(v_3)\cap E_2(v_3)} |W_1(v_3,c) - W_2(v_3,c)| \\
&= [W_1(v_1,v_3) + 0 + |W_1(v_1,v_2) - W_2(v_1,v_2)|] + [0 + W_2(v_2,v_1) + W_2(v_2,v_3) + 0] \\
&+ [0 + 0 + |W_1(v_3,v_1) - W_2(v_3,v_1)| + |W_1(v_3,v_2) - W_2(v_3,v_2)|] \\
&= [3 + 0 + 1] + [0 + 5 + 3 + 0] + [0 + 0 + 5 + 3] = 20.
\end{aligned}
$$

Hence we have the result that the distance for G_1 *and* G_2 *is* 20 *by metric* d_1.

Claim 1. *For all* $a \in V$, *one has*

$$
\begin{aligned}
&[(E_1(a) \triangle E_2(a)) \cup (E_1(a) \cap E_2(a))] \cup \\
&[(E_2(a) \triangle E_3(a)) \cup (E_2(a) \cap E_3(a))] \supseteq \\
&[(E_1(a) \triangle E_3(a)) \cup (E_1(a) \cap E_3(a))].
\end{aligned}
$$

Proof. It follows immediately from the fact that

$$(E_1(a) \cup E_2(a)) \cup (E_2(a) \cup E_3(a)) \supseteq (E_1(a) \cup E_3(a)).$$

□

Claim 2. *(semi-metric)*

1. $d_1(G_1, G_2) \geq 0$;
2. $d_1(G_1, G_2) = d_1(G_2, G_1)$;
3. $d_1(G_1, G_2) = 0$ *iff* $G_1 = G_2$.

Proof. By the definition, the first and second statements follow immediately. Here we show the third statement. Suppose $G_1 = G_2$. Then,

$$d_1(G_1, G_2) = \sum_{a\in V} [\sum_{c\in E_1(a)\cap E_2(a)} |W_1(a,c) - W_2(a,c)|] = 0.$$

On the other hand, if $d_1(G_1, G_2) = 0$, then $\forall a \in V[E_1(a) = E_2(a)]$ and $W_1 = W_2$, i.e., $G_1 = G_2$. □

From above inferences, if one allows $W(v,w) = 0$ for some $v \neq w$, then $d_1(G_1, G_2) = 0 \Rightarrow G_1 = G_2$ might not hold in some particular G_1 and G_2. Similarly, if one allows $W(v,v) = 0$, then for all $v \in V[(v,v) \in E]$ is a requirement.

Theorem 1. $(GG(V), =, d_1)$ *is a metric space.*

Proof. Since we have shown in Claim (2) that d_1 is a semi-metric, it suffices to show d_1 satisfies the triangle property:

$$d_1(G_1, G_2) + d_1(G_2, G_3) \geq d_1(G_1, G_3). \tag{2}$$

On the basis of Claim (1), we have the following inferences. Let $a \in V$ be arbitrary. Firstly, if $c \in E_1(a) - E_3(a)$, then $c \in E_1(a) - E_2(a)$ or $c \in E_1(a) \cap E_2(a)$. If $c \in E_1(a) - E_2(a)$, then it preserves the inequality of Equation (2) . If $c \in E_1(a) \cap E_2(a)$, then $c \in E_2(a) - E_3(a)$, i.e.,

$$c \in E_1(a) \cap E_2(a) \text{ and } c \in E_2(a) - E_3(a).$$

It follows that

$$|W_1(a,c) - W_2(a,c)| + W_2(a,c) \geq W_1(a,c),$$

i.e., the inequality of Equation (2) is preserved. Secondly, if $c \in E_3(a) - E_1(a)$, by the same analogy, the inequality of Equation (2) is also preserved. Lastly, if $c \in E_1(a) \cap E_3(a)$, then [$c \in E_1(a) - E_2(a)$ or $c \in E_1(a) \cap E_2(a)$] and [$c \in E_3(a) - E_2(a)$ or $c \in E_2(a) \cap E_3(a)$], i.e.,

$$c \in E_1(a) - E_2(a) \text{ and } c \in E_3(a) - E_2(a)$$

or

$$c \in E_1(a) \cap E_2(a) \text{ and } c \in E_2(a) \cap E_3(a).$$

It follows that

$$W_1(a,c) + W_3(a,c) \geq |W_1(a,c) - W_3(a,c)|,$$

$$|W_1(a,c) - W_2(a,c)| + |W_2(a,c) - W_3(a,c)| \geq |W_1(a,c) - W_3(a,c)|.$$

Hence, we have shown

$$\begin{aligned}
&\sum_{c \in E_1(a) - E_2(a)} W_1(a,c) + \sum_{c \in E_2(a) - E_1(a)} W_2(a,c) + \sum_{c \in E_1(a) \cap E_2(a)} |W_1(a,c) - W_2(a,c)| \\
&+ \sum_{c \in E_2(a) - E_3(a)} W_2(a,c) + \sum_{c \in E_3(a) - E_2(a)} W_3(a,c) + \sum_{c \in E_2(a) \cap E_3(a)} |W_2(a,c) - W_3(a,c)| \\
&\geq \sum_{c \in E_1(a) - E_3(a)} W_1(a,c) + \sum_{c \in E_3(a) - E_1(a)} W_3(a,c) + \sum_{c \in E_1(a) \cap E_3(a)} |W_1(a,c) - W_3(a,c)|,
\end{aligned}$$

and this completes our proof of Equation (2). □

4. Metric for Unlabelled Graphs

In this section, we show how to define a distance between graphs with unlabelled vertices. Let V^- be a set of distinct unlabelled vertices with $|V^-| = n$. Let $GG(V^-)$ be the set of generalized graphs whose vertex set is V^-. First of all, we show how to formalize unlabelled graphs. Let $M = \{m_1, m_2, ..., m_n\}$ be a set of dummy vertices for V^-. Then, each $G \in GG(V^-)$ could be modeled via this set of dummy vertices as $G^* = (M, E_M, W_M)$. Let $G_1^* = (M, E_M^1, W_M^1), G_2^* = (M, E_M^2, W_M^2) \in GG(V^-)$ be arbitrary. Let $N = \{v_1, v_2, ..., v_n\}$ be a set of names. Now fix the domain M and assign each dummy vertex a name via a naming function $\rho : M \to N$. Let $M \to N$ denote the set of all the naming functions. Now each unlabelled graph G could be formalized via naming functions as follows:

$$G = \{(\rho(M), E_{\rho(M)}, W_{\rho(M)}) : \rho \in M \to N\},$$

where $\rho(M) = \{\rho(m) : m \in M\}$; $E_{\rho(M)}$ and $W_{\rho(M)}$ denote the named edges and weights via ρ for E_M and W_M, respectively. G_1 and G_2 could be formalized as

$$G_1 = \{\{(\rho(M), E^1_{\rho(M)}, W^1_{\rho(M)})\} : \rho \in M \to N\},$$

$$G_2 = \{\{(\rho(M), E^2_{\rho(M)}, W^2_{\rho(M)})\} : \rho \in M \to N\}.$$

Since the modeling of unlabelled graph is not unique, we define an equivalence relation on $GG(V^-)$.

Definition 3. *$G_1 \equiv G_2$ iff $\exists \rho_1, \rho_2 \in M \to N$ such that $(\rho_1(M), E^1_{\rho_1(M)}, W^1_{\rho_1(M)}) = (\rho_2(M), E^2_{\rho_2(M)}, W^2_{\rho_2(M)})$.*

Example 2. *Let $M = \{m_1, m_2, m_3\}, N = \{v_1, v_2, v_3\}$. Then, (in a corresponding form) $M \to N$ consists of*

$$\rho_1 \equiv \begin{bmatrix} m_1 & m_2 & m_3 \\ v_1 & v_2 & v_3 \end{bmatrix}, \rho_2 \equiv \begin{bmatrix} m_1 & m_2 & m_3 \\ v_1 & v_3 & v_2 \end{bmatrix}, \rho_3 \equiv \begin{bmatrix} m_1 & m_2 & m_3 \\ v_2 & v_1 & v_3 \end{bmatrix},$$
$$\rho_4 \equiv \begin{bmatrix} m_1 & m_2 & m_3 \\ v_2 & v_3 & v_1 \end{bmatrix}, \rho_5 \equiv \begin{bmatrix} m_1 & m_2 & m_3 \\ v_3 & v_1 & v_2 \end{bmatrix}, \rho_6 \equiv \begin{bmatrix} m_1 & m_2 & m_3 \\ v_3 & v_2 & v_1 \end{bmatrix}.$$

Suppose $G_1^* = (M_1, E_{M_1}, W_{M_1}), G_2^* = (M_2, E_{M_2}, W_{M_2})$, where

$$\begin{aligned} &E_{M_1} = \{(m_1, m_2), (m_2, m_1), (m_2, m_3), (m_3, m_1)\} \\ &E_{M_2} = \{(m_2, m_3), (m_3, m_2), (m_3, m_1), (m_1, m_2)\} \\ &W_{M_1}(m_1, m_2) = 8, W_{M_1}(m_2, m_1) = 3, W_{M_1}(m_2, m_3) = 2, W_{M_1}(m_3, m_1) = 4 \\ &W_{M_2}(m_2, m_3) = 8, W_{M_2}(m_3, m_2) = 3, W_{M_2}(m_3, m_1) = 2, W_{M_2}(m_1, m_2) = 4 \end{aligned}$$

Hence G_1 consists of the following elements:

- $$(\rho_1(M) = \{v_1, v_2, v_3\}, E^1_{\rho_1(M)} = \{(v_1, v_2), (v_2, v_1), (v_2, v_3), (v_3, v_1)\},$$
$$W^1_{\rho_1(M)} = \{((v_1, v_2), 8), ((v_2, v_1), 3), ((v_2, v_3), 2), ((v_3, v_1), 4)\})$$
- $$(\rho_2(M) = \{v_1, v_2, v_3\}, E^1_{\rho_2(M)} = \{(v_1, v_3), (v_3, v_1), (v_3, v_2), (v_2, v_1)\},$$
$$W^1_{\rho_2(M)} = \{((v_1, v_3), 8), ((v_3, v_1), 3), ((v_3, v_2), 2), ((v_2, v_1), 4)\})$$
- $$(\rho_3(M) = \{v_1, v_2, v_3\}, E^1_{\rho_3(M)} = \{(v_2, v_1), (v_1, v_2), (v_1, v_3), (v_3, v_2)\},$$
$$W^1_{\rho_3(M)} = \{((v_2, v_1), 8), ((v_1, v_2), 3), ((v_1, v_3), 2), ((v_3, v_2), 4)\})$$
- $$(\rho_4(M) = \{v_1, v_2, v_3\}, E^1_{\rho_4(M)} = \{(v_2, v_3), (v_3, v_2), (v_3, v_1), (v_1, v_2)\},$$
$$W^1_{\rho_4(M)} = \{((v_2, v_3), 8), ((v_3, v_2), 3), ((v_3, v_1), 2), ((v_1, v_2), 4)\})$$
- $$(\rho_5(M) = \{v_1, v_2, v_3\}, E^1_{\rho_5(M)} = \{(v_3, v_2), (v_2, v_3), (v_2, v_1), (v_1, v_3)\},$$
$$W^1_{\rho_5(M)} = \{((v_3, v_2), 8), ((v_2, v_3), 3), ((v_2, v_1), 2), ((v_1, v_3), 4)\})$$
- $$(\rho_6(M) = \{v_1, v_2, v_3\}, E^1_{\rho_6(M)} = \{(v_3, v_2), (v_2, v_3), (v_2, v_1), (v_1, v_3)\},$$
$$W^1_{\rho_6(M)} = \{((v_3, v_2), 8), ((v_2, v_3), 3), ((v_2, v_1), 2), ((v_1, v_3), 4)\})$$

Similarly, one could list all the graphs in G_2, in particular,

$$(\rho_3(M) = \{v_1, v_2, v_3\}, E^2_{\rho_3(M)} = \{(v_1, v_3), (v_3, v_1), (v_3, v_2), (v_2, v_1)\},$$
$$W^2_{\rho_3(M)} = \{((v_1, v_3), 8), ((v_3, v_1), 3), ((v_3, v_2), 2), ((v_2, v_1), 4)\}).$$

Therefore,

$$\begin{aligned}
&(\rho_2(M) = \{v_1, v_2, v_3\}, E^1_{\rho_2(M)} = \{(v_1, v_3), (v_3, v_1), (v_3, v_2), (v_2, v_1)\},\\
&W^1_{\rho_2(M)} = \{((v_1, v_3), 8), ((v_3, v_1), 3), ((v_3, v_2), 2), ((v_2, v_1), 4)\})\\
&= (\rho_3(M) = \{v_1, v_2, v_3\}, E^2_{\rho_3(M)} = \{(v_1, v_3), (v_3, v_1), (v_3, v_2), (v_2, v_1)\},\\
&W^2_{\rho_3(M)} = \{((v_1, v_3), 8), ((v_3, v_1), 3), ((v_3, v_2), 2), ((v_2, v_1), 4)\}),
\end{aligned}$$

i.e., $G_1 \equiv G_2$.

Claim 3. $\equiv$ *is an equivalence relation on* $GG(V^-)$.

Proof. The result follows immediately from the definition. □

Definition 4. *(distance function: single edge, unlabelled) Define* $d_2 : GG(V^-) \times GG(V^-) \to \mathbb{R}^+$ *by*

$$\begin{aligned}
&d_2(G_1, G_2)\\
&:= min\{d_1((\rho(M), E^1_{\rho(M)}, W^1_{\rho(M)}), (\eta(M), E^2_{\eta(M)}, W^2_{\eta(M)})) : \rho, \eta \in M \to N\}.
\end{aligned} \tag{3}$$

It is obvious that if $G_1 \equiv G_2$, then $d_2(G_1, G_2) = 0$. Let us look a simple example that G_1 is not equivalent to G_2 in the following.

Example 3. *Let* $M = \{m_1, m_2, m_3\}$. *Suppose* $G_1^* = (M, E_{M_1}, W_{M_1}), G_2^* = (M, E_{M_2}, W_{M_2})$, *where*

$$\begin{aligned}
&E_{M_1} = \{(m_1, m_2), (m_1, m_3)\}\\
&E_{M_2} = \{(m_1, m_3), (m_1, m_2), (m_2, m_3)\}\\
&W_{M_1}(m_1, m_2) = 4, W_{M_1}(m_1, m_3) = 7\\
&W_{M_2}(m_1, m_3) = 3, W_{M_2}(m_1, m_2) = 8, W_{M_2}(m_2, m_3) = 1
\end{aligned}$$

Following the same procedures in Example (2), we could gain all the elements of G_1 *and* G_2. *By measuring the distances of their respective pairs (there are 36 pairs), and by Equation (4), one has the minimal one* $d_2(G_1, G_2) = d_1((M, E^1_{\rho_{i(M)}}, W^1_{\rho_{i(M)}}), (M, E^2_{\rho_{j(M)}}, W^2_{\rho_{j(M)}})) = |4-3| + |7-8| + 1 = 3$, *where* $\rho_i = \begin{bmatrix} m_1 & m_2 & m_3 \\ v_1 & v_2 & v_3 \end{bmatrix}$ *and* $\rho_j = \begin{bmatrix} m_1 & m_2 & m_3 \\ v_1 & v_3 & v_2 \end{bmatrix}$.

Claim 4. d_2 *is a semi-metric.*

Proof. It is obvious that $d_2(G, G) \geq 0$ and $d_2(G_1, G_2) = d_2(G_2, G_1)$. Suppose $G_1 \equiv G_2$. Then, there exist $\rho_1, \rho_2 \in M \to N$ such that $E^1_{\rho_1 M} = E^2_{\rho_2 M}$ and $W^1_{\rho_1 M} = W^2_{\rho_2 M}$, i.e., $d_2(G_1, G_2) = 0$. On the other hand, suppose $d_2(G_1, G_2) = 0$. Then, there exist $\rho_1, \rho_2 \in M \to N$ such that

$$d_1((\rho_1(M), E^1_{\rho_1(M)}, W^1_{\rho_1 M}), (\rho_2(M), E^2_{\rho_2(M)}, W^2_{\rho_2 M})) = 0,$$

i.e., $(\rho_1(M), E^1_{\rho_1(M)}, W^1_{\rho_1 M}) = (\rho_2(M), E^2_{\rho_2(M)}, W^2_{\rho_2 M})$, i.e., $G_1 \equiv G_2$. □

Claim 5.
$$d_1((\rho(M), E^1_{\rho(M)}, W^1_{\rho(M)}), (\eta(M), E^2_{\eta(M)}, W^2_{\eta(M)}))$$
$$= d_1((\zeta \circ \rho(M), E^1_{\zeta\circ\rho(M)}, W^1_{\zeta\circ\rho(M)}), (\zeta \circ \eta(M), E^2_{\zeta\circ\eta(M)}, W^2_{\zeta\circ\eta(M)}). \quad (4)$$

for all bijective function $\zeta \in N \to N$.

Proof. It suffices to show

$$d_1((V, E_1, W_1), (V, E_2, W_2)) = d_1((V, E^1_{\zeta(V)}, W^1_{\zeta(V)}), (V, E^2_{\zeta(V)}, W^2_{\zeta(V)})$$

for all $\zeta \in N \to N$, where $E^1_{\zeta(V)}$ denotes the relabelled edges via ζ of E_1 and $W^1_{\zeta(V)}$ denotes the weight function over the relabelled edges $E^1_{\zeta(V)}$.

Let $a \in V$ be arbitrary. Suppose

$$E_1(a) = \{a^1_1, a^1_2, ..., a^1_{k_1}\},$$

$$E_2(a) = \{a^2_1, a^2_2, ..., a^2_{k_2}\}.$$

Then,

$$E_1(\zeta(a)) = \{\zeta(a^1_1), \zeta(a^1_2), ..., \zeta(a^1_{k_1})\},$$
$$E_2(\zeta(a)) = \{\zeta(a^2_1), \zeta(a^2_2), ..., \zeta(a^2_{k_2})\}.$$

Hence, one has

$$\sum_{c\in E_1(a)-E_2(a)} W_1(a,c) + \sum_{c\in E_2(a)-E_1(a)} W_2(a,c) + \sum_{c\in E_1(a)\cap E_2(a)} |W_1(a,c) - W_2(a,c)|$$

$$= \sum_{c\in E_1(\zeta a)-E_2(\zeta a)} W_1(\zeta a, \zeta c) + \sum_{c\in E_2(\zeta a)-E_1(\zeta a)} W_2(\zeta a, \zeta c)$$
$$+ \sum_{c\in E_1(\zeta a)\cap E_2(\zeta a)} |W_1(\zeta a, \zeta c) - W_2(\zeta a, \zeta c)|,$$

where ζa denotes $\zeta(a)$ and ζc denotes $\zeta(c)$. Hence, we have shown

$$d_1((V, E_1, W_1), (V, E_2, W_2)) = d_1((V, E^1_{\zeta(V)}, W^1_{\zeta(V)}), (V, E^2_{\zeta(V)}, W^2_{\zeta(V)}).$$

□

Theorem 2. *$(GG(V^-), \equiv, d_2)$ is a metric space.*

Proof. Owing to Claim (4), it suffices to show the triangle transitivity property holds.

$$d_2(G_1, G_2) + d_2(G_2, G_3)$$
$$= d_1((\rho_1 M, E^1_{\rho_1 M}, W^1_{\rho_1 M}), (\rho_2 M, E^2_{\rho_2 M}, W^2_{\rho_2 M}))$$
$$+ d_1((\rho_3 M, E^2_{\rho_3 M}, W^2_{\rho_3 M}), (\rho_4 M, E^3_{\rho_4 M}, W^3_{\rho_4 M})),$$

where $\rho_j M$ denotes $\rho_j(M)$. Then, by Claim (5), one has

$$\begin{aligned}
& d_2(G_1, G_2) + d_2(G_2, G_3) \\
& = d_1((\rho_1 M, E^1_{\rho_1 M}, W^1_{\rho_1 M}), (\rho_2 M, E^2_{\rho_2 M}, W^2_{\rho_2 M})) \\
& + d_1((\rho_2 M, E^2_{\rho_2 M}, W^2_{\rho_2 M}), (\zeta \circ \rho_4 M, E^3_{\zeta\circ\rho_4 M}, W^3_{\zeta\circ\rho_4 M})) \\
& = d_1((\rho_1 M, E^1_{\rho_1 M}, W^1_{\rho_1 M}), (\rho_2 M, E^2_{\rho_2 M}, W^2_{\rho_2 M})) \\
& + d_1((\rho_2 M, E^2_{\rho_2 M}, W^2_{\rho_2 M}), (\rho_5 M, E^3_{\rho_5 M}, W^3_{\rho_5 M})) \\
& \geq d_1((\rho_1 M, E^1_{\rho_1 M}, W^1_{\rho_1 M}), (\rho_5 M, E^3_{\rho_5 M}, W^3_{\rho_5 M})) \\
& \geq d_2(G_1, G_3).
\end{aligned}$$

where $\zeta \in N \to$ is the bijective function satisfying $\zeta \circ \rho_3 = \rho_2$ and where $\rho_5 = \zeta \circ \rho_4$. $\square$

5. Computations

In this section, we show how to implement the above-mentioned metrics. Suppose $V = \{v_1, v_2, v_3, v_4\}$. To begin with, we implement d_1. Let e_{ij} denote the edge from node i to node j.

5.1. Labelled Vertices with Single Directed Edge

Given the two graphs G_1 and G_2 in Figure 1 and their respective adjacent matrices, in which the symbol ∞ (represented by a sufficient large real number) denotes there is no connection between the two nodes and represents a predetermined sufficiently large real number, in Table 1 (a pair α, β denote the weights of the directed edges e_{ij} and e_{ji}, respectively, where $i < j$).

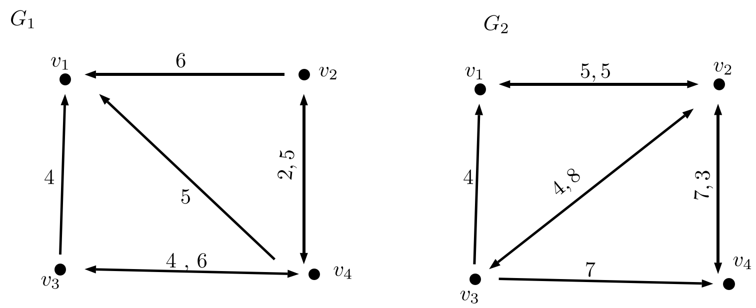

Figure 1. Labelled graphs: G_1 and G_2.

Table 1. Adjacent Matrices for G_1 and G_2.

$$W_1 = \begin{bmatrix} 0 & \infty & \infty & \infty \\ 6 & 0 & \infty & 2 \\ 4 & \infty & 0 & 4 \\ 5 & 5 & 6 & 0 \end{bmatrix}, W_2 = \begin{bmatrix} 0 & 5 & \infty & \infty \\ 5 & 0 & 4 & 7 \\ 4 & 8 & 0 & 7 \\ \infty & 3 & \infty & 0 \end{bmatrix}.$$

One obtains $E_1(v_1) = \{v_1\}, E_1(v_2) = \{v_1, v_2, v_4\}, E_1(v_3) = \{v_1, v_3, v_4\}, E_1(v_4) = \{v_1, v_2, v_3, v_4\}$; moreover, one also obtains $E_2(v_1) = \{v_1, v_2\}$, $E_2(v_2) = \{v_1, v_2, v_3, v_4\}$, $E_2(v_3) = \{v_1, v_2, v_3, v_4\}$, $E_2(v_4) = \{v_2, v_4\}$. The representation of these graphs via partial functions could be demonstrated by Table 2.

Table 2. Representing Directed Graphs via Partial Functions.

V	$W_1(.)$	$W_2(.)$
v_1	$\{(v_1,0)\}$	$\{(v_1,0),(v_2,5)\}$
v_2	$\{(v_1,6),(v_2,0),(v_4,2)\}$	$\{(v_1,5),(v_2,0),(v_3,4),(v_4,7)\}$
v_3	$\{(v_1,4),(v_3,0),(v_4,4)\}$	$\{(v_1,4),(v_2,8),(v_3,0),(v_4,7)\}$
v_4	$\{(v_1,5),(v_2,5),(v_3,6),(v_4,0)\}$	$\{(v_2,3),(v_4,0)\}$

By Equation (1), one has $d_1(G_1,G_2) = 5 + [|6-5| + |2-7| + 4] + [|4-4| + |4-7| + 8] + [5+6+|5-3|] = 39$. To simplify the whole computation, alternatively, this distance could also be obtained via the following matrix representation of Equation (1) and computation.

Definition 5.
$$\delta^n_{ij} = \begin{cases} 1, & \text{if } W_n(i,j) \neq \infty, \\ 0, & \text{otherwise}, \end{cases}$$
where $n \in \{1,2\}$.

Definition 6. *(distance between edges) Define each element e_{ij} of the distance matrix $[W_1,W_2]$ between W_1 and W_2 by*
$$e_{ij} = \delta^1_{ij} \cdot \delta^2_{ij} \cdot |e^1_{ij} - e^2_{ij}| + (1-\delta^1_{ij}) \cdot \delta^2_{ij} \cdot e^2_{ij} + \delta^1_{ij} \cdot (1-\delta^2_{ij}) \cdot e^1_{ij}.$$
where e^1_{ij} and e^2_{ij} denote element of i'th row, j'th column in W_1 and W_2, respectively.

On the basis of this definition, one has
$$[W_1,W_2] = \begin{bmatrix} 0 & 5 & 0 & 0 \\ 1 & 0 & 4 & 5 \\ 0 & 8 & 0 & 3 \\ 5 & 2 & 6 & 0 \end{bmatrix}.$$

Definition 7. *For any square matrix $S = (s_{ij})$, define $||S|| = \sum_{i,j=1}^{|S|} s_{ij}$.*

Then, Equation (1) could be represented and computed via the following matrix operation:
$$d_1(G_1,G_2) := ||[W_1,W_2]|| = 39.$$

5.2. Unlabelled Vertices with Singled Directed Edge

In this section, we show how to implement d_2 defined in Definition (4). Assume V is unlabelled. G_1 and G_2 are shown in Figure 2.

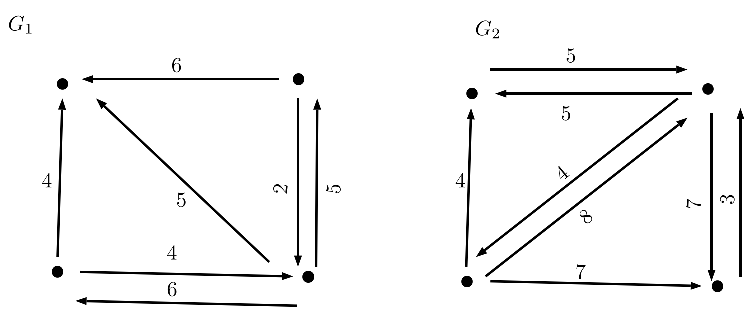

Figure 2. Unlabelled graphs: G_1 and G_2.

Let $N = \{v_1, v_2, v_3, v_4\}$ be the arbitrary names of the vertices of both G_1 and G_2. Let $PERM(n)$ denote all the permutations of the identity matrix with dimension n. By Equation (3), the distance between two unlabelled graphs could be represented and computed via the following matrix operations:

$$d_2(G_1, G_2) := min\{||[W_1, P^t W_2 P]|| : P \in PERM(|N|)\}, \tag{5}$$

where each P^t represents the transpose of the permutation matrix P. By computation, we have $|N| = 4$, $|PERM(|N|)| = 24$ and the distances between W_1 and each permutation of W_2 are listed as follows:

$$\{||[W_1, P^t W_2 P]|| : P \in PERM(4)\} =$$
$$\{39, 31, 41, 29, 27, 27, 43, 29, 53, 29, 35, 27, 45, 33, 57, 31, 35, 23, 43, 43, 53, 33, 49, 33\}.$$

Among which, the optimal permutation matrix is $\bar{P} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ and thus $d_2(G_1, G_2) = ||[W_1, \bar{P}^t W_2 \bar{P}]|| = 23$, where $\bar{P}^t W_2 \bar{P} = \begin{bmatrix} 0 & \infty & \infty & 3 \\ 7 & 0 & 4 & 8 \\ \infty & \infty & 0 & 5 \\ 7 & 4 & 5 & 0 \end{bmatrix}$. The corresponding minimal pair of graphs could be shown in Figure 3.

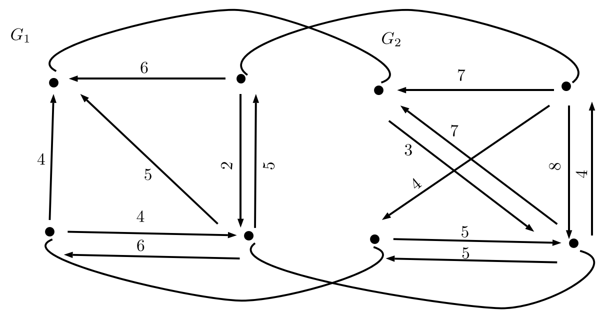

Figure 3. Optimal pair of graphs: G_1 and G_2.

This could be interpreted as the complexity of the overlap of these two graphs based on corresponding vertices, i.e., this overlap yields the minimal complexity of the graphs.

6. Conclusions

In this article, we have shown how to define distances between graphs over either a set of labelled or unlabelled vertices via metrics d_1 and d_2, respectively. We also give a computational approaches to implement the computation of d_1 and d_2 via adjacent matrix operations. This implementation gives an efficient and fast computation of the distance between any two such graphs. This type of distance could then be applied in measuring the distance between networks or tree structures.

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