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A New Generalized Definition of Fractal–Fractional Derivative with Some Applications

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Abstract: In this study, a new generalized fractal–fractional (FF) derivative is proposed. By applying this definition to some elementary functions, we show its compatibility with the results of the FF derivative in the Caputo sense with the power law. The main elements of classical differential calculus are introduced in terms of this new derivative. Thus, we establish and demonstrate the basic operations with derivatives, chain rule, mean value theorems with their immediate applications and inverse function’s derivative. We complete the theory of generalized FF calculus by proposing a notion of integration and presenting two important results of integral calculus: the fundamental theorem and Barrow’s rule. Finally, we analytically solve interesting FF ordinary differential equations by applying our proposed definition.

Keywords: fractal–fractional differentiation; fractal–fractional integration; fractal–fractional derivative in Caputo sense; fractal–fractional ordinary differential equations



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1. Introduction

The theory of fractional calculus has interested many researchers, as a theoretical extension of classical mathematical analysis that has also been successfully applied to solve important problems in different scientific disciplines. In the evolution of the definition of fractional derivatives, two formulations have been proposed: a non-local approach and another based on a local conception. The non-local concept of a fractional derivative, which has played a fundamental role in the development of fractional calculus, includes well-known definitions such as the Riemann–Liouville (RL) and Caputo (C) derivatives. The key properties and applications of these definitions are mentioned in [1–3].

Classical fractional derivatives, such as C and RL derivatives, have both advantages and disadvantages. However, in these definitions of derivatives, the linearity property is satisfied, and they do not possess certain essential properties of the ordinary derivative. For instance, the non-zero RL derivative of a constant differs from the behavior of the ordinary derivative. Furthermore, these definitions lack the fundamental properties of the ordinary derivative, including product, quotient, and chain rules. However, the C derivative is only defined for differentiable functions.

The local formulation of non-integer order derivatives arises with the idea of overcoming the disadvantages associated with non-local fractional derivatives. The definitions of local fractional derivatives are established based on incremental ratios. A well-known definition in this category is the conformable derivative, introduced by Khalil et al. [4]. This definition successfully addresses some of the limitations of non-local fractional derivatives. Furthermore, conformable calculus offers a direct approach for obtaining analytical solutions for various applications of fractional calculus problems. However, according

to [5], the conformable derivative has a disadvantage, and its results cannot be considered satisfactory in comparison to the C definition for certain functions. Recently, in [6], a new generalized derivative of non-integer order, the so-called Abu-Shady–Kaabar derivative, allows fractional differential equations to be solved analytically in a simple way, whose results are in exact agreement with those obtained through the RL or C derivative. Furthermore, the study of this new derivative has been extended to important fields of classical analysis, such as special functions or the fixed-point theorem [7,8].

On the other hand, another notable non-integer local differential operator is the so-called fractal derivative introduced in [9]. This derivative is established from the fractal definition, and it has been used to model various scientific phenomena concerning power law scaling, such as quantum mechanics and turbulence [9,10].

Some recent research studies have proven the internal connection between fractional calculus and fractal calculus in the context of geometry where the fractal dimension of the function with a fractional order can be modified via the change in fractional calculus formulation [11] (see also [12] for more recent research about the connection between them via the approximation of continuous functions).

In [13], a hybrid approach of differentiation that combines both fractional and fractal differentiations is introduced. In this new differential operator, various properties (memory effect, heterogeneity, elastic viscosity, and fractal geometry) of the dynamic system are considered.

Based on the existing literature and the limitations of non-local fractional derivatives, our research aims to introduce a new local FF derivative and explore its fundamental properties. To accomplish this objective, we structured our study into several stages, which are outlined as follows:

1. The definition of the generalized FF derivative of order α is introduced in this study, which produces results consistent with the outcomes obtained using the FF derivative of order α in the C sense with power law, as mentioned in [13].
2. Furthermore, we establish the fundamental elements of generalized FF calculus, including operations with generalized FF α, γ -differentiable functions, the chain rule, mean value theorems, and the inverse function theorem.
3. Then, we define the generalized fractal α, γ -integral and present two significant results of integral calculus, namely the fundamental theorem of calculus and Barrow's rule, in this context.
4. Finally, we give some interesting applications of the proposed derivative to FF ordinary differential equations.
5. Our results are original and novel because they provide a simple mathematical tool that can be applied efficiently in modeling various systems and phenomena, proposed with fractional order and fractal dimension, in sciences, engineering, economics, and medicine, where the connection between both fractional calculus and fractal geometry can play an important role in studying those systems.

2. Preliminaries

The fundamental notions are established, which will be necessary for the development of our research. Thus, the FF derivative of a function in the C sense with power law is defined as [13]:

Definition 1. Assume that $f(t)$ is differentiable on interval $[a, \infty)$, with $a \geq 0$; if f is fractal differentiable on $[a, \infty)$ with order γ , then the FF derivative of f of order α in C sense with power law is written as:

$${}^{\text{FF}}D_t^{\alpha, \gamma} f(t) = \frac{1}{\Gamma(n - \alpha)} \int_a^t (t - \tau)^{n - \alpha - 1} \frac{df(\tau)}{d\tau^\gamma} d\tau, \quad n - 1 < \alpha, \gamma \leq n, \quad n \in \mathbb{N}, \quad (1)$$

where

$$\frac{df(\tau)}{d\tau^\gamma} = \lim_{t \rightarrow \tau} \frac{f(t) - f(\tau)}{t^\gamma - \tau^\gamma}. \quad (2)$$

Remark 1. Note that $\frac{df(\tau)}{d\tau^\gamma}$ given in Equation (2) is the fractal derivative of order γ , with $\gamma > 0$, introduced in [9,10].

Remark 2. In particular, if $a = 0$ and $n = 1$, Equation (1) reduces to the following equation

$${}_0^{FF}D_t^{\alpha,\gamma}f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \frac{df(\tau)}{d\tau^\gamma} d\tau. \quad (3)$$

Remark 3. It is easy to prove that the fractal derivative of a function can be written in terms of the classical derivative as follows:

$$\frac{df(\tau)}{d\tau^\gamma} = \frac{1}{\gamma} \tau^{1-\gamma} \frac{df(\tau)}{d\tau}. \quad (4)$$

Now, we establish two interesting results on the FF derivative of order α defined in Equation (3), which will be useful in the developments that we include in the next section.

Theorem 1. Let $0 < \alpha, \gamma < 1$, and $\lambda > -1$. Then, we have:

$${}_0^{FF}D_t^{\alpha,\gamma}(t^\lambda) = \frac{\lambda \Gamma(\lambda - \gamma + 1)}{\gamma \Gamma(\lambda - \alpha - \gamma + 2)} t^{\lambda - \alpha - \gamma + 1}. \quad (5)$$

Proof. Using the classic definitions of the alpha and beta functions, Equations (3) and (4), and the change of variables $\tau = tu$, we obtain

$$\begin{aligned} {}_0^{FF}D_t^{\alpha,\gamma}(t^\lambda) &= \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \frac{d(t^\lambda)}{d\tau^\gamma} d\tau = \frac{1}{\gamma \Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \tau^{1-\gamma} \frac{d(t^\lambda)}{d\tau} d\tau = \\ &= \frac{\lambda t^{\lambda-\alpha-\gamma+1}}{\gamma \Gamma(1-\alpha)} \int_0^1 u^{\lambda-\gamma} (1-u)^{-\alpha} du = \frac{\lambda}{\gamma \Gamma(1-\alpha)} \beta(\lambda - \gamma + 1, 1 - \alpha) t^{\lambda-\alpha-\gamma+1} = \\ &= \frac{\lambda}{\beta \Gamma(1-\alpha)} \frac{\Gamma(\lambda - \gamma + 1) \Gamma(1-\alpha)}{\Gamma(\lambda - \alpha - \gamma + 2)} t^{\lambda-\alpha-\gamma+1} = \frac{\lambda \Gamma(\lambda - \gamma + 1)}{\gamma \Gamma(\lambda - \alpha - \gamma + 2)} t^{\lambda-\alpha-\gamma+1}. \end{aligned}$$

□

Remark 4. Note that if $f(t) = c$ for every real constant c , then ${}_0^{FF}D_t^{\alpha,\gamma}(c) = 0$.

Theorem 2. Let $0 < \alpha, \gamma < 1$. Suppose that a function $f(t)$ analytic at the origin with McLaurin expansion given by:

$$f(t) = \sum_{k=0}^{\infty} a_k t^k, \quad (6)$$

for $t \in [0, r)$ with $r \in \mathbb{R}^+$. Then, we have:

$${}_0^{FF}D_t^{\alpha,\gamma}f(t) = \sum_{k=0}^{\infty} a_k {}_0^{FF}D_t^{\alpha,\gamma}(t^k). \quad (7)$$

Proof. Using Equation (3) to the function $f(t)$ with the series expansion, we have:

$$\begin{aligned} {}_0^{FF}D_t^{\alpha,\gamma}f(t) &= \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \frac{d}{d\tau^\gamma} \left(\sum_{k=0}^{\infty} a_k \tau^k \right) d\tau = \\ &= \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \frac{1}{\gamma} \tau^{1-\gamma} \frac{d}{d\tau} \left(\sum_{k=0}^{\infty} a_k \tau^k \right) d\tau. \end{aligned}$$

Since the power series converges uniformly on any closed interval $[0, \rho]$ with $0 < \rho < r$, we can integrate term by term in the above equation [14]. Thus,

$$\begin{aligned} {}_0^{FF}D_t^{\alpha, \gamma} f(t) &= \sum_{k=0}^{\infty} a_k \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \frac{1}{\gamma} \tau^{1-\gamma} \frac{d(\tau^k)}{d\tau} d\tau = \\ \sum_{k=0}^{\infty} a_k \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \frac{d(\tau^k)}{d\tau} d\tau &= \sum_{k=0}^{\infty} a_k {}_0^{FF}D_t^{\alpha, \gamma} (t^k). \end{aligned}$$

□

Remark 5. From Theorem 1, Equation (7) can be expressed as:

$${}_0^{FF}D_t^{\alpha, \gamma} f(t) = \sum_{k=1}^{\infty} a_k \frac{k\Gamma(k-\gamma+1)}{\gamma\Gamma(k-\alpha-\gamma+2)} t^{k-\alpha-\gamma+1}. \quad (8)$$

3. Generalized Fractal–Fractional Derivative and Its Properties

A new local type of FF derivative is discussed in this section based on Definition 1 and Theorems 1 and 2, established in the previous section. Likewise, we present and prove the main properties of this proposed derivative.

Definition 2. For function $f: [0, \infty) \rightarrow \mathbb{R}$, the generalized FF (GFF) derivative of order $0 < \alpha \leq 1$, of f at $t > 0$ is written as:

$${}^{GFF}D^{\alpha, \gamma} f(t) = \lim_{\varepsilon \rightarrow 0} \frac{f(t + \varepsilon M(\alpha, \gamma, \lambda) t^{2-\alpha-\gamma}) - f(t)}{\varepsilon}, \quad (9)$$

where $M(\alpha, \gamma, \lambda) = \frac{\Gamma(\lambda-\gamma+1)}{\gamma\Gamma(\lambda-\alpha-\gamma+2)}$ with $0 < \gamma \leq 1$ and $\lambda > -1$.

If f is GFF α, γ -differentiable in some $(0, a)$, $a > 0$, and $\lim_{t \rightarrow 0^+} {}^{GFF}D^{\alpha, \gamma} f(t)$ exists, then it is expressed as:

$${}^{GFF}D^{\alpha, \gamma} f(0) = \lim_{t \rightarrow 0^+} {}^{GFF}D^{\alpha, \gamma} f(t). \quad (10)$$

Remark 6. It is interesting to highlight some special cases of Definition 2:

- (i) If $\alpha = \gamma = 1$, then Equation (9) becomes the classical definition of the derivative.
- (ii) If $\gamma = 1$, then Equation (9) becomes the definition of Abu-Shady–Kaabar fractional derivative of order α proposed in [6].
- (iii) If $\alpha = 1$, then Equation (9) becomes the definition of fractal derivative of order γ introduced in [9,10].

Theorem 3. Let $0 < \alpha, \gamma \leq 1$, and let f be generalized fractal α, γ -differentiable at a point $t > 0$. If, additionally, f is differentiable function, then

$${}^{GFF}D^{\alpha, \gamma} f(t) = M(\alpha, \gamma, \lambda) t^{2-\alpha-\gamma} \frac{df(t)}{dt}, \quad (11)$$

where $M(\alpha, \gamma, \lambda) = \frac{\Gamma(\lambda-\gamma+1)}{\gamma\Gamma(\lambda-\alpha-\gamma+2)}$ with $\lambda > -1$.

Proof. Let $h = \varepsilon M(\alpha, \gamma, \lambda) t^{2-\alpha-\gamma}$ in Equation (9), and then $\varepsilon = \frac{h}{M(\alpha, \gamma, \lambda) t^{\alpha+\gamma-2}}$. Therefore,

$$\begin{aligned} {}^{GFF}D^{\alpha, \gamma} f(t) &= \lim_{h \rightarrow 0} \frac{f(t+h) - f(t)}{\frac{h}{M(\alpha, \gamma, \lambda) t^{\alpha+\gamma-2}}} = M(\alpha, \gamma, \lambda) t^{2-\alpha-\gamma} \lim_{h \rightarrow 0} \frac{f(t+h) - f(t)}{h} = \\ &= M(\alpha, \gamma, \lambda) t^{2-\alpha-\gamma} \frac{df(t)}{dt}. \end{aligned}$$

□

Remark 7. Note that if $0 < \alpha \leq 1$, $\gamma = 1$, and $\lambda > -1$, then Equation (9) can be written as:

$${}^{GFF}D^{\alpha,\gamma}f(t) = M(\alpha, 1, \lambda)t^{1-\alpha}\frac{df(t)}{dt} = M(\alpha, 1, \lambda)T_{\alpha}f(t),$$

where T_{α} is the conformable derivative of order α introduced in [4].

Remark 8. Consider a function $f(t) = t^{\lambda}$, $\lambda > -1$. Using Theorem 3, the following result is easily obtained,

$$\begin{aligned} {}^{GFF}D^{\alpha,\gamma}(t^{\lambda}) &= M(\alpha, \gamma, \lambda)t^{2-\alpha-\gamma}\lambda t^{\lambda-1} \\ &= \frac{\lambda\Gamma(\lambda-\gamma+1)}{\gamma\Gamma(\lambda-\alpha-\gamma+2)}t^{\lambda-\alpha-\gamma+1}. \end{aligned} \quad (12)$$

Note that the above result is compatible with the result of the FF derivative of order α in the C sense with power law expressed in Equation (1).

Theorem 4. If a function $f : [0, \infty) \rightarrow \mathbb{R}$ is GFF α, γ -differentiable at t_0 , $0 < \alpha, \gamma \leq 1$, $\lambda > -1$, then f is continuous at t_0 .

Proof. Since

$$f(t_0 + \varepsilon M(\alpha, \gamma, \lambda)t_0^{2-\alpha-\gamma}) - f(t_0) = \frac{f(t_0 + \varepsilon M(\alpha, \gamma, \lambda)t_0^{2-\alpha-\gamma}) - f(t_0)}{\varepsilon}\varepsilon,$$

Then,

$$\lim_{\varepsilon \rightarrow 0} [f(t_0 + \varepsilon M(\alpha, \gamma, \lambda)t_0^{2-\alpha-\gamma}) - f(t_0)] = \lim_{\varepsilon \rightarrow 0} \frac{f(t_0 + \varepsilon M(\alpha, \gamma, \lambda)t_0^{2-\alpha-\gamma}) - f(t_0)}{\varepsilon} \cdot \lim_{\varepsilon \rightarrow 0} \varepsilon,$$

Let $h = \varepsilon M(\alpha, \gamma, \lambda)t_0^{2-\alpha-\gamma}$. Then,

$$\lim_{h \rightarrow 0} f(t_0 + h) - f(t_0) = 0,$$

which implies that

$$\lim_{h \rightarrow 0} f(t_0 + h) = f(t_0),$$

Hence, f is continuous at t_0 . $\square$

Theorem 5. Let $0 < \alpha, \gamma \leq 1$, $\lambda > -1$ and let f, g be GFF α, γ -differentiable at a point $t > 0$. Then, we have:

- (i) ${}^{GFF}D^{\alpha,\gamma}(af + bg)(t) = a {}^{GFF}D^{\alpha,\gamma}f(t) + b {}^{GFF}D^{\alpha,\gamma}g(t)$, $\forall a, b \in \mathbb{R}$.
- (ii) ${}^{GFF}D^{\alpha,\gamma}(\mu) = 0$, $\forall$ constant functions $f(t) = \mu$.
- (iii) ${}^{GFF}D^{\alpha,\gamma}(fg)(t) = f(t){}^{GFF}D^{\alpha,\gamma}g(t) + g(t){}^{GFF}D^{\alpha,\gamma}f(t)$.
- (iv) ${}^{GFF}D^{\alpha,\gamma}\left(\frac{f}{g}\right)(t) = \frac{g(t){}^{GFF}D^{\alpha,\gamma}f(t) - f(t){}^{GFF}D^{\alpha,\gamma}g(t)}{[g(t)]^2}$.

Proof. Parts (i) and (ii) are followed directly from the mentioned definition. Let us only show (iii) since it is important. Now, for fixed $t > 0$,

$$\begin{aligned} {}^{GFF}D^{\alpha,\gamma}[fg](t) &= \lim_{\varepsilon \rightarrow 0} \frac{f(t+\varepsilon M(\alpha,\gamma,\lambda)t^{2-\alpha-\gamma})g(t+\varepsilon M(\alpha,\gamma,\lambda)t^{2-\alpha-\gamma}) - f(t)g(t)}{\varepsilon} \\ &= \lim_{\varepsilon \rightarrow 0} \frac{f(t+\varepsilon M(\alpha,\gamma,\lambda)t^{2-\alpha-\gamma})g(t+\varepsilon M(\alpha,\gamma,\lambda)t^{2-\alpha-\gamma}) - f(t+\varepsilon M(\alpha,\gamma,\lambda)t^{2-\alpha-\gamma})g(t)}{\varepsilon} \\ &\quad + \lim_{\varepsilon \rightarrow 0} \frac{f(t+\varepsilon M(\alpha,\gamma,\lambda)t^{2-\alpha-\gamma})g(t) - f(t)g(t)}{\varepsilon} \\ &= \lim_{\varepsilon \rightarrow 0} \left[f(t+\varepsilon M(\alpha,\gamma,\lambda)t^{2-\alpha-\gamma}) \cdot \frac{g(t+\varepsilon M(\alpha,\gamma,\lambda)t^{2-\alpha-\gamma}) - g(t)}{\varepsilon} \right] + g(t) \\ &\quad \cdot \lim_{\varepsilon \rightarrow 0} \frac{f(t+\varepsilon M(\alpha,\gamma,\lambda)t^{2-\alpha-\gamma}) - f(t)}{\varepsilon} \\ &= \lim_{\varepsilon \rightarrow 0} f(t+\varepsilon M(\alpha,\gamma,\lambda)t^{2-\alpha-\gamma}) \cdot {}^{GFF}D^{\alpha,\gamma}g(t) + g(t) \cdot {}^{GFF}D^{\alpha,\gamma}f(t), \end{aligned}$$

Since f is continuous at t , $\lim_{\varepsilon \rightarrow 0} f(t+\varepsilon M(\alpha,\gamma,\lambda)t^{2-\alpha-\gamma}) = f(t)$, this completes the proof of parts (iii) and (iv), which can be proven in a similar way. $\square$

Remark 9. Now, we show that the results obtained by applying this derivative proposed for certain elementary functions are compatible with the results of the FF derivative of order α in the C sense with power law as in Equation (1).

(i) Exponential function $f(t) = e^{\gamma t}$, $\gamma \in \mathbb{C}$.

Using the fact that $e^{\gamma t} = \sum_{k=0}^{\infty} \frac{\gamma^k}{k!} t^k$, we have

$${}^{GFF}D^{\alpha,\gamma}(e^{\gamma t}) = \sum_{k=0}^{\infty} \frac{\gamma^k}{k!} {}^{GFF}D^{\alpha,\gamma}(t^k),$$

From Remark 8, we get:

$${}^{GFF}D^{\alpha,\gamma}(t^k) = {}_0^F D^{\alpha,\gamma}(t^k)$$

Finally, the following result is obtained,

$${}^{GFF}D^{\alpha,\gamma}(e^{\gamma t}) = {}_0^F D^{\alpha,\gamma}(e^{\gamma t}), \quad (13)$$

(ii) Sine function $f(t) = \sin \vartheta t$.

Using the fact that $\sin \vartheta t = \frac{1}{2i}(e^{i\vartheta t} - e^{-i\vartheta t})$, we have:

$${}^{GFF}D^{\alpha,\gamma}(\sin \vartheta t) = \frac{1}{2i} \left({}^{GFF}D^{\alpha,\gamma}(e^{i\vartheta t}) - {}^{GFF}D^{\alpha,\gamma}(e^{-i\vartheta t}) \right),$$

From (13), we obtain:

$${}^{GFF}D^{\alpha,\gamma}(\sin \vartheta t) = \frac{1}{2i} \left({}_0^F D^{\alpha,\gamma}(e^{i\vartheta t}) - {}_0^F D^{\alpha,\gamma}(e^{-i\vartheta t}) \right) = {}_0^F D^{\alpha,\gamma} \left(\frac{1}{2i}(e^{i\vartheta t} - e^{-i\vartheta t}) \right),$$

Finally, the following result is obtained:

$${}^{GFF}D^{\alpha,\gamma}(\sin \vartheta t) = {}_0^F D^{\alpha,\gamma}(\sin \vartheta t), \quad (14)$$

(iii) Cosine function $f(t) = \cos \vartheta t$.

Using the fact that $\cos \vartheta t = \frac{1}{2}(e^{i\vartheta t} + e^{-i\vartheta t})$, we have:

$${}^{GFF}D^{\alpha,\gamma}(\cos \vartheta t) = \frac{1}{2} \left({}^{GFF}D^{\alpha,\gamma}(e^{i\vartheta t}) + {}^{GFF}D^{\alpha,\gamma}(e^{-i\vartheta t}) \right),$$

From (13), we obtain:

$${}^{GFF}D^{\alpha,\gamma}(\cos \theta t) = \frac{1}{2} \left({}^{FF}D^{\alpha,\gamma} \left(e^{i\theta t} \right) + {}^{FF}D^{\alpha,\gamma} \left(e^{-i\theta t} \right) \right) = {}^{FF}D^{\alpha,\gamma} \left(\frac{1}{2} \left(e^{i\theta t} + e^{-i\theta t} \right) \right),$$

Finally, the following result is obtained,

$${}^{GFF}D^{\alpha,\gamma}(\cos \theta t) = {}^{GFF}D^{\alpha,\gamma}(\cos \theta t). \quad (15)$$

Now, we establish a fundamental result of classical mathematical analysis, the chain rule, in the context of FF calculus. Note that this extension is possible due to the local character of the proposed GFF derivative.

Theorem 6 (Chain Rule). Let $0 < \alpha, \gamma \leq 1, \lambda > -1$, g GFF α, γ -differentiable at $t > 0$ and f is differentiable at $g(t)$, then

$${}^{GFF}D^{\alpha,\gamma}(f \circ g)(t) = f'(g(t)) {}^{GFF}D^{\alpha,\gamma}g(t), \quad (16)$$

Proof. We show the result via the standard limit approach. If the function g is constant in a neighbourhood of $a > 0$, then ${}^{GFF}D^{\alpha,\gamma}(f \circ g)(t) = 0$. If g is not constant in a neighbourhood of $a > 0$, we can find a $t_0 > 0$, such that $g(t_1) \neq g(t_2)$ for any $t_1, t_2 \in (a - t_0, a + t_0)$. Now, since g is continuous at a , for a sufficiently small ϵ , we have:

$$\begin{aligned} {}^{GFF}D^{\alpha,\gamma}(f \circ g)(t)(a) &= \lim_{\epsilon \rightarrow 0} \frac{f(g(a + \epsilon M(\alpha, \gamma, \lambda)a^{2-\alpha-\gamma})) - f(g(a))}{\epsilon} = \\ \lim_{\epsilon \rightarrow 0} \left[\frac{f(g(a + \epsilon M(\alpha, \gamma, \lambda)a^{2-\alpha-\gamma})) - f(g(a))}{g(a + \epsilon M(\alpha, \gamma, \lambda)a^{2-\alpha-\gamma}) - g(a)} \cdot \frac{g(a + \epsilon M(\alpha, \gamma, \lambda)a^{2-\alpha-\gamma}) - g(a)}{\epsilon} \right] &= \\ \lim_{\epsilon \rightarrow 0} \frac{f(g(a + \epsilon M(\alpha, \gamma, \lambda)a^{2-\alpha-\gamma})) - f(g(a))}{g(a + \epsilon M(\alpha, \gamma, \lambda)a^{2-\alpha-\gamma}) - g(a)} \cdot \lim_{\epsilon \rightarrow 0} \frac{g(a + \epsilon M(\alpha, \gamma, \lambda)a^{2-\alpha-\gamma}) - g(a)}{\epsilon}, \end{aligned}$$

Taking

$$\epsilon_0 = g(a + \epsilon M(\alpha, \gamma, \lambda)a^{2-\alpha-\gamma}) - g(a).$$

In the first factor, we have:

$$\lim_{\epsilon \rightarrow 0} \frac{f(g(a + \epsilon M(\alpha, \gamma, \lambda)a^{2-\alpha-\gamma})) - f(g(a))}{g(a + \epsilon M(\alpha, \gamma, \lambda)a^{2-\alpha-\gamma}) - g(a)} = \lim_{\epsilon_0 \rightarrow 0} \frac{f(g(a) + \epsilon_0) - f(g(a))}{\epsilon_0},$$

And from here

$$\begin{aligned} {}^{GFF}D^{\alpha,\gamma}(f \circ g)(t)(a) &= \lim_{\epsilon_0 \rightarrow 0} \frac{f(g(a) + \epsilon_0) - f(g(a))}{\epsilon_0} \cdot \lim_{\epsilon \rightarrow 0} \frac{g(a + \epsilon M(\alpha, \gamma, \lambda)a^{2-\alpha-\gamma}) - g(a)}{\epsilon} \\ &= f'(g(a)) {}^{GFF}D^{\alpha,\gamma}g(a). \end{aligned}$$

The proof is completed. $\square$

Remark 10. From the result above, it is easy to obtain the GFF derivative of order α of the following elementary functions:

- (i) ${}^{GFF}D^{\alpha,\gamma} \left(\frac{\gamma}{(\alpha+\gamma-1)\Gamma(\alpha)} t^{\alpha+\gamma-1} \right) = 1,$
- (ii) ${}^{GFF}D^{\alpha,\gamma} \left(e^{\frac{\gamma}{(\alpha+\gamma-1)\Gamma(\alpha)} t^{\alpha+\gamma-1}} \right) = e^{\frac{\gamma}{(\alpha+\gamma-1)\Gamma(\alpha)} t^{\alpha+\gamma-1}},$
- (iii) ${}^{GFF}D^{\alpha,\gamma} \left(\sin \left(\frac{\gamma}{(\alpha+\gamma-1)\Gamma(\alpha)} t^{\alpha+\gamma-1} \right) \right) = \cos \left(\frac{\gamma}{(\alpha+\gamma-1)\Gamma(\alpha)} t^{\alpha+\gamma-1} \right),$
- (iv) ${}^{GFF}D^{\alpha,\gamma} \left(\cos \left(\frac{\gamma}{(\alpha+\gamma-1)\Gamma(\alpha)} t^{\alpha+\gamma-1} \right) \right) = -\sin \left(\frac{\gamma}{(\alpha+\gamma-1)\Gamma(\alpha)} t^{\alpha+\gamma-1} \right),$

Remark 11. Using the fact that differentiability implies GFF α, γ -differentiability and assuming $g(t) > 0$, Equation (16) can be re-written as:

$$\frac{1}{M(\alpha, \gamma, \lambda)} g(t)^{\alpha+\gamma-2} {}^{GFF}D^{\alpha, \gamma} f(g(t)) {}^{GFF}D^{\alpha, \gamma} g(t), \quad (17)$$

where $M(\alpha, \gamma, \lambda) = \frac{\Gamma(\lambda-\gamma+1)}{\gamma\Gamma(\lambda-\alpha-\gamma+2)}$ with $\lambda > -1$.

The extension of the mean value theorems of classical mathematical analysis was also the subject of our research. Thus, we establish these theorems for GFF differentiable functions and discuss some interesting consequences.

Theorem 7 (Roll's theorem for GFF α, γ -differentiable functions). Let $a > 0, 0 < \alpha, \gamma \leq 1, \lambda > -1$ and $f : [a, b] \rightarrow R$ be a given function that satisfies:

- (i) f is continuous on $[a, b]$,
- (ii) f is GFF α, γ -differentiable on (a, b) ,
- (iii) $f(a) = f(b)$.

Then, there exists $c \in (a, b)$, such that ${}^{GFF}D^{\alpha, \gamma} f(c) = 0$.

Proof. Since f is continuous on $[a, b]$ and $f(a) = f(b)$, there is $c \in (a, b)$, which is a point of local extrema. With no loss of generality, assume c is a point of local minimum. So

$${}^{GFF}D^{\alpha, \gamma} f(c) = \lim_{\varepsilon \rightarrow 0^+} \frac{f(t + \varepsilon M(\alpha, \gamma, \lambda)t^{2-\alpha-\gamma}) - f(t)}{\varepsilon} = \lim_{\varepsilon \rightarrow 0^-} \frac{f(t + \varepsilon M(\alpha, \gamma, \lambda)t^{2-\alpha-\gamma}) - f(t)}{\varepsilon},$$

But, the first limit is non-negative, and the second limit is non-positive. Hence, ${}^{GFF}D^{\alpha, \gamma} f(c) = 0$. $\square$

Theorem 8 (mean value theorem for GFF α, γ -differentiable functions). Let $a > 0, 0 < \alpha, \gamma \leq 1$, and $f : [a, b] \rightarrow R$ be a given function that satisfies:

- (i) f is continuous on $[a, b]$,
- (ii) f is local GFF α, γ -differentiable on (a, b) ,

Then, there exists $c \in (a, b)$, such that ${}^{GFF}D^{\alpha, \gamma} f(c) = \frac{f(b)-f(a)}{\frac{\gamma}{(\alpha+\gamma-1)\Gamma(\alpha)}(b^{\alpha+\gamma-1}-a^{\alpha+\gamma-1})}$.

Proof. Consider the function,

$$g(t) = f(t) - f(a) - \left(\frac{f(b)-f(a)}{\frac{\gamma}{(\alpha+\gamma-1)\Gamma(\alpha)}(b^{\alpha+\gamma-1}-a^{\alpha+\gamma-1})} \right) \left(\frac{\gamma}{(\alpha+\gamma-1)\Gamma(\alpha)}(t^{\alpha+\gamma-1}-a^{\alpha+\gamma-1}) \right),$$

Then

$${}^{GFF}D^{\alpha, \gamma} g(t) = {}^{GF/C}D^{\alpha} f(t) - \left(\frac{f(b)-f(a)}{\frac{\gamma}{(\alpha+\gamma-1)\Gamma(\alpha)}(b^{\alpha+\gamma-1}-a^{\alpha+\gamma-1})} \right) {}^{GFF}D^{\alpha, \gamma} \left(\frac{\gamma t^{\alpha+\gamma-1}}{(\alpha+\gamma-1)\Gamma(\alpha)} \right),$$

From Remark 10, we obtain:

$${}^{GFF}D^{\alpha, \gamma} g(t) = {}^{GFF}D^{\alpha, \gamma} f(t) - \frac{f(b)-f(a)}{\frac{\gamma}{(\alpha+\gamma-1)\Gamma(\alpha)}(b^{\alpha+\gamma-1}-a^{\alpha+\gamma-1})},$$

At $c \in [a, b]$.

$${}^{GFF}D^{\alpha,\gamma}g(c) = {}^{GFF}D^{\alpha,\gamma}f(c) - \frac{f(b) - f(a)}{\frac{\gamma}{(\alpha+\gamma-1)\Gamma(\alpha)}(b^{\alpha+\gamma-1} - a^{\alpha+\gamma-1})},$$

And the auxiliary function $g(t)$ satisfies all conditions of Theorem 7. Hence, there exists $c \in (a, b)$, such that ${}^{GFF}D^{\alpha,\gamma}g(c) = 0$. Then, we obtain:

$${}^{GFF}D^{\alpha,\gamma}f(c) = \frac{f(b) - f(a)}{\frac{\gamma}{(\alpha+\gamma-1)\Gamma(\alpha)}(b^{\alpha+\gamma-1} - a^{\alpha+\gamma-1})},$$

□

Theorem 9. Let $a > 0$, $0 < \alpha, \gamma \leq 1$, and $f : [a, b] \rightarrow \mathbb{R}$ be a given function that satisfies:

- (i) f is continuous on $[a, b]$,
- (ii) f is GFF α, γ -differentiable on (a, b) ,

If ${}^{GFFP}D^{\alpha,\gamma}f(t) = 0$, for all $t \in (a, b)$, then, f is constant on $[a, b]$.

Proof. Suppose ${}^{GFF}D^{\alpha,\gamma}f(t) = 0$ for all $t \in (a, b)$. Let $t_1, t_2 \in [a, b]$ with $t_1 < t_2$. So, the closed interval $[t_1, t_2]$ is contained in $[a, b]$, and the open interval (t_1, t_2) is contained in (a, b) . Hence, f is continuous on $[t_1, t_2]$ and generalized fractal α, γ -differentiable on (t_1, t_2) . So, from Theorem 8, there exists $c \in (t_1, t_2)$ with

$$\frac{f((t_2)) - f((t_1))}{\frac{\gamma}{(\alpha+\gamma-1)\Gamma(\alpha)}((t_2)^{\alpha+\gamma-1} - (t_1)^{\alpha+\gamma-1})} = {}^{GFF}D^{\alpha,\gamma}f(c) = 0,$$

Therefore, $f(t_2) - f(t_1) = 0$ and $f(t_2) = f(t_1)$. Since t_1 and t_2 are arbitrary numbers in $[a, b]$ with $t_2 > t_1$, f is constant on $[a, b]$. □

Corollary 1. Let $a > 0$, $0 < \alpha, \gamma \leq 1$, and $F, G : [a, b] \rightarrow \mathbb{R}$ be functions such that ${}^{GFF}D^{\alpha,\gamma}F(t) = {}^{GFF}D^{\alpha,\gamma}G(t)$ for all $t \in (a, b)$. Then, there exists a constant C such that

$$F(t) = G(t) + C,$$

Proof. By simply applying the above theorem to $H(t) = F(t) - G(t)$, it can be proven easily. □

Theorem 10. Let $a > 0$, $0 < \alpha, \gamma \leq 1$, and $f : [a, b] \rightarrow \mathbb{R}$ be a given function that satisfies:

- (i) f is continuous on $[a, b]$,
- (ii) f is GFF α, γ -differentiable on (a, b) .

Then, we have the following:

1. If ${}^{GFFP}D^{\alpha,\gamma}f(t) > 0$, for all $t \in (a, b)$, then f is increasing on $[a, b]$.
2. If ${}^{GFFP}D^{\alpha,\gamma}f(t) < 0$, for all $t \in (a, b)$, then f is decreasing on $[a, b]$.

Proof. Similarly, using Theorem 9's proof, there exists $c \in (t_1, t_2)$ with

$$\frac{f((t_2)) - f((t_1))}{\frac{\gamma}{(\alpha+\gamma-1)\Gamma(\alpha)}((t_2)^{\alpha+\gamma-1} - (t_1)^{\alpha+\gamma-1})} = {}^{GFF}D^{\alpha,\gamma}f(c),$$

1. If ${}^{GFF}D^{\alpha,\gamma}f(t) > 0$, then $f(t_2) > f(t_1)$ for $t_2 > t_1$. Therefore, f is strictly increasing on $[a, b]$, since t_1 and t_2 are arbitrary numbers of $[a, b]$.

2. If ${}^{GFF}D^{\alpha,\gamma}f(t) < 0$, then $f(t_2) < f(t_1)$ for $t_2 > t_1$. Therefore, f is strictly decreasing on $[a, b]$, since t_1 and t_2 are arbitrary numbers of $[a, b]$. $\square$

Theorem 11 (racetrack-type principal). Let $a > 0$, $0 < \alpha, \gamma \leq 1$, and $f, g : [a, b] \rightarrow \mathbb{R}$ be the given functions that satisfies:

- (i) f and g are continuous on $[a, b]$,
- (ii) f and g are GFF α, γ -differentiable on (a, b) ,
- (iii) ${}^{GFF}D^{\alpha,\gamma}f(t) \leq {}^{GFF}D^{\alpha,\gamma}g(t)$ for all $t \in (a, b)$.

Then, we have the following:

- 1. If $f(a) = g(a)$, then $f(t) \leq g(t)$, for all $t \in [a, b]$.
- 2. If $f(b) = g(b)$, then $f(t) \geq g(t)$, for all $t \in [a, b]$.

Proof. Consider $h(t) = g(t) - f(t)$. Then, h is continuous on $[a, b]$ and GFF α, γ -differentiable on (a, b) . Also, using the linearity of ${}^{GFF}D^{\alpha,\gamma}$ and the fact that ${}^{GFF}D^{\alpha,\gamma}f(t) \leq {}^{GFF}D^{\alpha,\gamma}g(t)$ for all $t \in (a, b)$, we obtain ${}^{GFF}D^{\alpha,\gamma}h(t) \geq 0$, for all $t \in (a, b)$. So, through Theorem 10, h is increasing (non-decreasing). Hence, for any $a \leq t \leq b$, we have $h(a) \leq h(t)$. Since $h(a) = g(a) - f(a) = 0$ by the assumption, the result follows. Similarly, for part 2 of Theorem 11, since for any $a \leq t \leq b$, we have $h(t) \leq h(b)$ and $h(b) = f(b) - g(b) = 0$, the result follows. $\square$

Theorem 12 (extended mean value theorem for GFF α, γ -differentiable functions). Let $a > 0$, $0 < \alpha, \gamma \leq 1$, $\lambda > -1$ and $f, g : [a, b] \rightarrow \mathbb{R}$ be the given functions that satisfies:

- (i) f, g are continuous on $[a, b]$,
- (ii) f, g are GFF α, γ -differentiable on (a, b) ,

Then, there exists $c \in (a, b)$, such that $\frac{{}^{GFF}D^{\alpha,\gamma}f(c)}{{}^{GFF}D^{\alpha,\gamma}g(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$.

Proof. Consider the function:

$$F(t) = f(t) - f(a) - \left(\frac{f(b) - f(a)}{g(b) - g(a)} \right) (g(t) - g(a)),$$

Then, the function F satisfies the conditions of Theorem 7. Hence, there exists $c \in (a, b)$, such that ${}^{GFF}D^{\alpha,\gamma}F(c) = 0$. Using the linearity of the GFF α, γ -derivative and the fact that ${}^{GFF}D^{\alpha,\gamma}F(c) = 0$, with c a constant, the result follows. $\square$

Remark 12. Another interesting result in the context of the proposed GFF calculus is a modified version of the mean value theorem for GFF α, γ -differentiable functions. Next, we will establish and prove this result.

Theorem 13 (modified value theorem for generalized fractal α, γ -differentiable functions). Let $a > 0$, $0 < \alpha, \gamma \leq 1$, and $f : [a, b] \rightarrow \mathbb{R}$ be a given function that satisfies:

- (i) f is continuous on $[a, b]$,
- (ii) f is GFF α, γ -differentiable on (a, b) .

Then, there exists $c \in (a, b)$, such that

$$\frac{{}^{GFF}D^{\alpha,\gamma}f(c)}{\frac{\Gamma(2-\gamma)}{\gamma\Gamma(3-\alpha-\gamma)}c^{2-\alpha-\gamma}} = \frac{f(b) - f(a)}{b - a},$$

Proof. Consider the function:

$$g(t) = f(t) - f(a) - \left(\frac{f(b) - f(a)}{b - a} \right) (t - a),$$

Then, the function g satisfies the conditions of Theorem 7. Hence, there exists $c \in (a, b)$, such that ${}^{GFF}D^{\alpha, \gamma}g(c) = 0$. Therefore,

$${}^{GFF}D^{\alpha, \gamma}g(c) = {}^{GFF}D^{\alpha, \gamma}f(c) - \left(\frac{f(b) - f(a)}{b - a} \right) \frac{\Gamma(2 - \gamma)}{\gamma\Gamma(3 - \alpha - \gamma)} c^{2 - \alpha - \gamma} = 0,$$

Hence,

$$\frac{{}^{GFF}D^{\alpha, \gamma}f(c)}{\frac{\Gamma(2 - \gamma)}{\gamma\Gamma(3 - \alpha - \gamma)} c^{2 - \alpha - \gamma}} = \frac{f(b) - f(a)}{b - a},$$

□

Remark 13. From the above theorem, we can easily establish similar consequences as those obtained in Theorem 8 (see Theorems 9–11 and Corollary 1).

Definition 3. Let $I \subset (0, \infty)$ an open interval, $0 < \alpha, \gamma \leq 1$, and $f : I \rightarrow \mathbb{R}$, we will say that f is of class $C^{\alpha, \gamma}$ on the interval I , which we write as $f \in C^{\alpha, \gamma}(I, \mathbb{R})$, if f is GFF α, γ -differentiable on I and GFF α, γ -derivative is continuous on I .

Theorem 14. Let $I \subset (0, \infty)$ an open interval, $0 < \alpha, \gamma \leq 1$, and $f : I \rightarrow \mathbb{R}$ be a function of class $C^{\alpha, \gamma}$ on the interval I . Suppose $f(a) = b$ for some $a \in I$, and ${}^{GFF}D^{\alpha, \gamma}f(a) \neq 0$. Then, there is an open neighborhood U of a in which f admits an inverse function f^{-1} of class $C^{\alpha, \gamma}$ on the open neighborhood $V = f(U)$ of b , and its GFF α, γ -derivative is:

$${}^{GFF}D^{\alpha, \gamma}f^{-1}(y) = \left(\frac{\Gamma(2 - \gamma)}{\gamma\Gamma(3 - \alpha - \gamma)} \right)^2 \frac{t^{\alpha + \gamma - 2} y^{\alpha + \gamma - 2}}{{}^{GFF}D^{\alpha, \gamma}f(t)}, \quad \forall y \in V, t = f^{-1}(y). \quad (18)$$

Proof. Since $f(t)$ is continuous in the open interval I , it is a known fact that there exists an open neighborhood U of a in which ${}^{GFF}D^{\alpha, \gamma}f(t)$ has a constant sign (the sign of ${}^{GFF}D^{\alpha, \gamma}f(a)$). From Remark 13, it follows f that is strictly monotonic on U (increasing if ${}^{GFF}D^{\alpha, \gamma}f(a) > 0$, decreasing if ${}^{GFF}D^{\alpha, \gamma}f(a) < 0$). Therefore, f is continuous and strictly monotonic on U , so there is the inverse function of the one-to-one function $f : U \rightarrow V$, with $V = f(U)$. This inverse $f^{-1} : V \rightarrow U$ is of class $C^{\alpha, \gamma}$ and strictly monotonic (in the same sense that f is) on V . Equation (18) can be easily obtained from the identity $f(f^{-1}(y)) = y$ for all $y \in V$, in which the GFF α, γ -derivative (with respect to y) is calculated, applying the chain rule as follows:

$$\frac{\gamma\Gamma(3 - \alpha - \gamma)}{\Gamma(2 - \gamma)} t^{\alpha + \gamma - 2} \cdot {}^{GFF}D^{\alpha, \gamma}f^{-1}(y) {}^{GFF}D^{\alpha, \gamma}f(t) = \frac{\Gamma(2 - \gamma)}{\gamma\Gamma(3 - \alpha - \gamma)} y^{2 - \alpha - \gamma}, \quad \forall y \in V, t = f^{-1}(y),$$

□

Finally, we present the following definition for the GFF α, γ -integral of a function f starting at $a \geq 0$:

Definition 4. ${}^{GFF}I_{a, \gamma}^{\lambda}(f)(t) = \frac{1}{M(\alpha, \gamma, \lambda)} \int_a^t \frac{f(x)}{x^{2 - \alpha - \gamma}} dx$, where this integral is basically the usual Riemann improper integral, $M(\alpha, \gamma, \lambda) = \frac{\Gamma(\lambda - \gamma + 1)}{\gamma\Gamma(\lambda - \alpha - \gamma + 2)}$, $0 < \alpha, \gamma \leq 1$, and $\lambda > -1$.

From the definition above, we can establish two important results:

Theorem 15. ${}^{GFF}D^{\alpha,\gamma} {}^{GFF}I_{\alpha,\gamma}^a(f)(t) = f(t)$, for $t \geq a$, where f is any continuous function in the domain of ${}^{GFF}I_{\alpha,\gamma}^a$.

Proof. Since f is continuous, then ${}^{GFF}I_{\alpha,\gamma}^a(f)(t)$ is differentiable. Hence,

$$\begin{aligned} {}^{GFF}D^{\alpha,\gamma} {}^{GFF}I_{\alpha,\gamma}^a(f)(t) &= M(\alpha, \gamma, \lambda) t^{2-\alpha-\gamma} \frac{d}{dt} \left({}^{GFF}I_{\alpha,\gamma}^a(f)(t) \right) \\ &= M(\alpha, \gamma, \lambda) t^{2-\alpha-\gamma} \frac{d}{dt} \left(\frac{1}{M(\alpha, \gamma, \lambda)} \int_a^t \frac{f(x)}{x^{2-\alpha-\gamma}} dx \right) = t^{2-\alpha-\gamma} \frac{f(t)}{t^{2-\alpha-\gamma}} = f(t), \end{aligned}$$

□

Theorem 16. Let $a > 0$, $0 < \alpha, \gamma \leq 1$, $\lambda > -1$, and f be a continuous real-valued function on interval $[a, b]$. Let G any real-valued function with the property ${}^{GFF}D^{\alpha,\gamma}G(t) = f(t)$ for all $t \in [a, b]$. Then

$${}^{GFF}I_{\alpha,\gamma}^a(f)(b) = G(b) - G(a), \quad (19)$$

Proof. First, let F be a function on $[a, b]$ defined as $F(t) = {}^{GFF}I_{\alpha,\gamma}^a(f)(t)$, which can be called GFF α, γ -integral function of f .

By using Theorem 15, ${}^{GFF}D^{\alpha,\gamma}F(t) = f(t)$ for all $t \in [a, b]$.

Since F and G have the same GFF α, γ -derivative, then by corollary 1, there exists a real constant C , such that $G(t) = F(t) + C$ for all $t \in [a, b]$.

Finally, $G(b) - G(a)$ is computed as follows:

$$\begin{aligned} G(b) - G(a) &= (F(b) + C) - (F(a) + C) \\ &= \frac{1}{M(\alpha, \gamma, \lambda)} \left(\int_a^b \frac{f(t)}{t^{2-\alpha-\gamma}} dt - \int_a^a \frac{f(t)}{t^{2-\alpha-\gamma}} dt \right) \\ &= \frac{1}{M(\alpha, \gamma, \lambda)} \int_a^b \frac{f(t)}{t^{2-\alpha-\gamma}} dt. \end{aligned}$$

□

4. Applications

In this section, we will solve several interesting FF ordinary differential equations in the sense of the proposed GFF derivative.

Example 1. Consider the initial value problem involving a GFF ordinary differential equation of order $\alpha = \frac{1}{2}, \gamma = \frac{1}{3}$ as follows:

$${}^{GFF}D^{\frac{1}{2}, \frac{1}{3}}y(t) = e^{-5t}, y(0) = 0, \quad (20)$$

To find the solution to the differential equation in Equation (20), we use the fact that $e^{-5t} = \sum_{k=0}^{\infty} (-1)^k \frac{5^k}{k!} t^k$ and apply Equation (11) to obtain:

$$\frac{3\Gamma(\lambda + \frac{2}{3})}{\Gamma(\lambda + \frac{7}{6})} t^{\frac{7}{6}} \frac{dy(t)}{dt} = \sum_{k=0}^{\infty} (-1)^k \frac{5^k}{k!} t^k,$$

If we rearrange the above equation and integrate on both sides, it follows:

$$y(t) = \frac{\Gamma(\lambda + \frac{7}{6})}{3\Gamma(\lambda + \frac{2}{3})} \sum_{k=0}^{\infty} (-1)^k \frac{5^k}{k!} \int t^{k-\frac{7}{6}} dt = \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(\lambda + \frac{7}{6})}{3\Gamma(\lambda + \frac{2}{3})} \frac{5^k}{k!} \frac{t^{k-\frac{1}{6}}}{k-\frac{1}{6}} + C,$$

By taking $\lambda = k - \frac{1}{6}$, we have:

$$y(t) = \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(k+1)}{3\Gamma(k+\frac{1}{2})} \frac{5^k t^{k-\frac{1}{6}}}{k! k - \frac{1}{6}} + C = \sum_{k=0}^{\infty} (-1)^k \frac{5^k}{3(k-\frac{1}{6})\Gamma(k+\frac{1}{2})} t^{k-\frac{1}{6}} + C.$$

Finally, using the initial condition $y(0) = 0$, we obtain:

$$y(t) = \sum_{k=0}^{\infty} (-1)^k \frac{5^k}{3(k-\frac{1}{6})\Gamma(k+\frac{1}{2})} t^{k-\frac{1}{6}}.$$

Note that this solution is consistent with the solution to the corresponding FF (in the sense of C) Initial Value Problem (IVP).

Example 2. Consider the IVP involving a GFF ordinary differential equation of order $\alpha = \frac{1}{3}$, $\gamma = \frac{1}{4}$ as follows:

$${}^{GFF}D^{\frac{1}{3}, \frac{1}{4}} y(t) = t^2 \sinh(4t), y(0) = 0, \quad (21)$$

To find the solution to the differential equation in Equation (21), we use the fact that $\sinh(4t) = \sum_{k=0}^{\infty} \frac{4^{2k+1}}{(2k+1)!} t^{2k+1}$ and apply Equation (11) to obtain:

$$\frac{4\Gamma(\lambda + \frac{3}{4})}{\Gamma(\lambda + \frac{17}{12})} t^{\frac{17}{12}} \frac{dy(t)}{dt} = \sum_{k=0}^{\infty} \frac{4^{2k+1}}{(2k+1)!} t^{2k+3},$$

If we rearrange the above equation and integrate on both sides, it follows:

$$y(t) = \frac{\Gamma(\lambda + \frac{17}{12})}{4\Gamma(\lambda + \frac{3}{4})} \sum_{k=0}^{\infty} \frac{4^{2k+1}}{(2k+1)!} \int t^{2k+\frac{19}{12}} dt = \sum_{k=0}^{\infty} \frac{\Gamma(\lambda + \frac{17}{12})}{4\Gamma(\lambda + \frac{3}{4})} \frac{4^{2k+1}}{(2k+1)!} \frac{t^{2k+\frac{31}{12}}}{2k+\frac{31}{12}} + C,$$

By taking $\lambda = 2k + \frac{31}{12}$, we have:

$$y(t) = \sum_{k=0}^{\infty} \frac{\Gamma(2k+4)4^{2k+1}}{4(2k+\frac{31}{12})(2k+1)!\Gamma(2k+\frac{10}{3})} t^{2k+\frac{31}{12}} + C,$$

Finally, using the initial condition $y(0) = 0$, we obtain:

$$y(t) = \sum_{k=0}^{\infty} \frac{\Gamma(2k+4)4^{2k+1}}{4(2k+\frac{31}{12})(2k+1)!\Gamma(2k+\frac{10}{3})} t^{2k+\frac{31}{12}}.$$

Note that this solution is consistent with the solution to the corresponding FF (in the sense of C) IVP.

Remark 14. In the following example, we will solve a generalized linear fractal–fractional differential equation, but, also, we will define this type of differential equation and prove a result in which its general solution is established.

Definition 5. The generalized linear FF differential equation of order α , γ is defined as

$${}^{GFF}D^{\alpha, \gamma} y(t) + p(t)y(t) = q(t), \quad (22)$$

where $0 < \alpha, \gamma \leq 1$, $\lambda > -1$, and p, q are real-valued continuous functions on an interval $I \subset [0, \infty]$.

Theorem 17. The general solution of the GFF differential Equation (22) is expressed by:

$$y(t) = e^{-GFF I_{\alpha,\gamma}(p)(t)} \left[GFF I_{\alpha,\gamma} \left(q(t) e^{GFF I_{\alpha,\gamma}(p)(t)} \right) + C \right], \quad (23)$$

where C is a real constant.

Proof. Using Theorem 3, Equation (22) can be expressed as:

$$y'(t) + \frac{\gamma \Gamma(\lambda - \alpha - \gamma + 2)}{\Gamma(\lambda - \gamma + 1)} \frac{p(t)}{t^{2-\alpha-\gamma}} y(t) = \frac{\gamma \Gamma(\lambda - \alpha - \gamma + 2)}{\Gamma(\lambda - \gamma + 1)} \frac{q(t)}{t^{2-\alpha-\gamma}}, \quad (24)$$

Since the above equation is a classical first-order linear differential equation, its general solution is written as:

$$y(t) = e^{-\frac{\gamma \Gamma(\lambda - \alpha - \gamma + 2)}{\Gamma(\lambda - \gamma + 1)} \int \frac{p(t)}{t^{2-\alpha-\gamma}} dt} \left[\frac{\gamma \Gamma(\lambda - \alpha - \gamma + 2)}{\Gamma(\lambda - \gamma + 1)} \int e^{\frac{\gamma \Gamma(\lambda - \alpha - \gamma + 2)}{\Gamma(\lambda - \gamma + 1)} \int \frac{p(t)}{t^{2-\alpha-\gamma}} dt} \frac{q(t)}{t^{2-\alpha-\gamma}} dt + C \right], \quad (25)$$

where C is a real constant. Finally, using Definition 5 and substituting into Equation (25), our result follows directly. $\square$

Now, we can solve an example that involves a generalized linear FF differential equation of order α, γ using the proposed method.

Example 3. Consider the generalized linear FF differential equation of order $\alpha = \frac{1}{2}, \gamma = \frac{1}{3}$ as follows:

$$GFF D^{\frac{1}{2}, \frac{1}{3}}(y)(t) + y(t) = t^3 + \frac{9\Gamma(\lambda + \frac{2}{3})}{\Gamma(\lambda + \frac{7}{6})} t^{\frac{19}{6}}, y(0) = 0. \quad (26)$$

Taking in Equation (26) $\alpha = \frac{1}{2}, \gamma = \frac{1}{3}, p(t) = 1$, and $q(t) = t^3 + \frac{9\Gamma(\lambda + \frac{2}{3})}{\Gamma(\lambda + \frac{7}{6})} t^{\frac{19}{6}}$, we have:

$$y(t) = e^{-\frac{\Gamma(\lambda + \frac{7}{6})}{3\Gamma(\lambda + \frac{2}{3})} \int t^{-\frac{7}{6}} dt} \left[\frac{\Gamma(\lambda + \frac{7}{6})}{3\Gamma(\lambda + \frac{2}{3})} \int \left(t^{\frac{11}{6}} + \frac{9\Gamma(\lambda + \frac{2}{3})}{\Gamma(\lambda + \frac{7}{6})} t^2 \right) e^{\frac{\Gamma(\lambda + \frac{7}{6})}{3\Gamma(\lambda + \frac{2}{3})} \int t^{-\frac{7}{6}} dt} dt + C \right],$$

where C is a real constant.

If in the above equation we calculate the integrals and simplify, we easily obtain:

$$y(t) = e^{\frac{2\Gamma(\lambda + \frac{7}{6})}{\Gamma(\lambda + \frac{2}{3})} t^{-\frac{1}{6}}} \left[t^3 e^{-\frac{2\Gamma(\lambda + \frac{7}{6})}{\Gamma(\lambda + \frac{2}{3})} t^{-\frac{1}{6}}} + C \right] = t^3 + C e^{\frac{2\Gamma(\lambda + \frac{7}{6})}{\Gamma(\lambda + \frac{2}{3})} t^{-\frac{1}{6}}},$$

Finally, the initial condition $y(0) = 0$ implies that $C = 0$. Hence, $y(t) = t^3$.

We finish this section by discussing another interesting differential equation in the sense of the GFF derivative, specifically, the generalized Bernoulli FF differential equation. As in the classic case, we propose to solve this equation by reducing it to a generalized linear FF differential equation. Thus, consider the generalized Bernoulli FF differential equation of non-integer order α , given by

$$GFF D^{\alpha,\gamma}(y)(t) + p(t)y(t) = q(t)y(t)^n, \quad (27)$$

where $0 < \alpha, \gamma \leq 1, \lambda > -1, n \neq 0, 1$, and p, q are real-valued continuous functions on an interval $I \subset [0, \infty]$.

Using Theorem 3, Equation (27) can be written as:

$$y'(t) + \frac{\gamma\Gamma(\lambda - \alpha - \gamma + 2)}{\Gamma(\lambda - \gamma + 1)} \frac{p(t)}{t^{2-\alpha-\gamma}} y(t) = \frac{\gamma\Gamma(\lambda - \alpha - \gamma + 2)}{\Gamma(\lambda - \gamma + 1)} \frac{q(t)}{t^{2-\alpha-\gamma}} y(t)^n.$$

The above equation, through the change of variable $z = y^{1-n}$, can be reduced to the following linear ordinary differential equation:

$$z'(t) + (1-n) \frac{\gamma\Gamma(\lambda - \alpha - \gamma + 2)}{\Gamma(\lambda - \gamma + 1)} \frac{p(t)}{t^{2-\alpha-\gamma}} z(t) = (1-n) \frac{\gamma\Gamma(\lambda - \alpha - \gamma + 2)}{\Gamma(\lambda - \gamma + 1)} \frac{q(t)}{t^{2-\alpha-\gamma}},$$

According to Theorem 15, the general solution of Equation (27) is given by:

$$y(t) = \left(e^{-(1-n)GFF I_{\alpha,\gamma} p(t)} \left[GFF I_{\alpha,\gamma} \left((1-n)q(t)e^{(1-n)GFF I_{\alpha,\gamma} p(t)} \right) + C \right] \right)^{\frac{1}{1-n}}, \quad (28)$$

In the following example, we apply this proposed method to solve a generalized Bernoulli FF differential equation.

Example 4. Consider the generalized Bernoulli FF differential equation of order $\alpha = \frac{1}{3}, \gamma = \frac{1}{4}$ as follows:

$${}^{GFF}D^{\frac{1}{3}, \frac{1}{4}} y(t) + \frac{4\Gamma(\lambda + \frac{5}{4})}{\Gamma(\lambda + \frac{17}{12})} t^{\frac{17}{12}} y(t) = t^{\frac{5}{3}} e^{-2t} y(t)^{-1}. \quad (29)$$

Taking in Equation (29) $\alpha = \frac{1}{3}, \gamma = \frac{1}{4}, n = -1, p(t) = \frac{4\Gamma(\lambda + \frac{5}{4})}{\Gamma(\lambda + \frac{17}{12})} t^{\frac{17}{12}}$, and $q(t) = t^{\frac{5}{3}} e^{-2t}$, we have:

$$y(t) = \left(e^{-\frac{\Gamma(\lambda + \frac{17}{12})}{2\Gamma(\lambda + \frac{5}{4})} \int \frac{4\Gamma(\lambda + \frac{5}{4})}{\Gamma(\lambda + \frac{17}{12})} dt} \left[\frac{\Gamma(\lambda + \frac{17}{12})}{4\Gamma(\lambda + \frac{5}{4})} \int 2t^{\frac{1}{4}} e^{-2t} e^{\frac{\Gamma(\lambda + \frac{17}{12})}{2\Gamma(\lambda + \frac{5}{4})} \int \frac{4\Gamma(\lambda + \frac{5}{4})}{\Gamma(\lambda + \frac{17}{12})} dt} dt + C \right] \right)^{\frac{1}{2}},$$

where C is a real constant.

If in the above equation we calculate the integrals and simplify, it follows directly:

$$y(t) = \left(e^{-2t} \left[\frac{2}{5} \frac{\Gamma(\lambda + \frac{17}{12})}{4\Gamma(\lambda + \frac{5}{4})} t^{\frac{5}{4}} + C \right] \right)^{\frac{1}{2}}.$$

5. Conclusions

This study introduces the concept of the GFF derivative of order α , which produces consistent results with the FF derivative of a function in the C sense with the power law when applied to elementary functions. The fundamental elements of GFF calculus, including operations with GFF α, γ -differentiable functions, chain rule, mean value theorems, and inverse function theorem, are established. Additionally, the generalized fractal α, γ -integral is defined, and two significant results of integral calculus, namely the fundamental theorem of calculus and Barrow's rule, are presented within this framework. We analytically solved interesting examples of FF ordinary differential equations with the help of the proposed definition of the GFF derivative, obtaining solutions that agree exactly with the results of the FF derivative of a function in the C sense with the power law. Our results allow us to conclude that this derivative, being of a local type, provides a simple tool to obtain analytical solutions to many natural science and engineering problems that present a fractal effect and that involve ordinary differential equations of non-integer order.

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