

Review

Euler's Equation of Continuity: Additional Terms of High Order of Smallness—An Overview

Vladislav M. Ovsyannikov

Department of Waterways, Ports and Hydraulic Structures, Russian University of Transport MIIT, 127994 Moscow, Russia; OvsyannikovVM@yandex.ru

Abstract: Professor N.E. Zhukovsky was a famous Russian mechanic and engineer. In 1876 he defended his master's thesis at Moscow University. At a careful reading of N.E. Zhukovsky's master's thesis in 1997, V.A. Bubnov—a professor at the Moscow City Pedagogical University—discovered terms of the second order of smallness in the continuity equation for an incompressible fluid. Zhukovsky calculated them, but did not use the amount of substance in the balance. Ten years later, the author found high-order terms in Euler's derivation of the 1752 continuity equation for an incompressible fluid. The physical meaning of the additional terms became clear after the derivation in 2006 of the continuity equation with terms of high order of smallness for a compressible gas. The higher order terms of the smallness of the continuity equation penetrate into the inhomogeneous part of the wave equation and lead to the generation of self-oscillations, vibrations, sound, and the initial stage of turbulent pulsations. The stochastic approach ensured success in modeling turbulent flows. The use of high-order terms of smallness of the Euler continuity equation makes it possible to transfer the description of some part of the motions from the stochastic part of the equation to the deterministic part. The article contains a review of works with the derivation of the inhomogeneous wave equation. These works use additional terms of a high order of smallness in the continuity equation.



Citation: Ovsyannikov, V.M. Euler's Equation of Continuity: Additional Terms of High Order of Smallness—An Overview. *Fluids* **2021**, *6*, 162. <https://doi.org/10.3390/fluids6040162>

Academic Editor: Artur V. Dmitrenko

Received: 21 February 2021

Accepted: 9 April 2021

Published: 17 April 2021

Publisher's Note: MDPI stays neutral with regard to jurisdictional claims in published maps and institutional affiliations.



Copyright: © 2021 by the author. Licensee MDPI, Basel, Switzerland. This article is an open access article distributed under the terms and conditions of the Creative Commons Attribution (CC BY) license (<https://creativecommons.org/licenses/by/4.0/>).

Keywords: stochastic equation; continuity equation; inhomogeneous wave equation; self-oscillation; vibration; sound

1. Introduction

Stochastic methods penetrate the mechanics of liquid and gas to describe the subtle physical processes of turbulence. The addition of stochastic terms to deterministic equations leads to the description of some physical effects. The high-order terms of the continuity equation found by N.E. Zhukovsky [1] and V.A. Bubnov [2] can be used in stochastic methods. Stochastic models are used to simulate the subtle effects of turbulence in homogeneous and heterogeneous flows. The theory of equivalence of measures of deterministic and fluctuation motions in turbulence is well-developed.

A new system of stochastic equations is presented, which was obtained on the basis of the equivalence relations of measures between the deterministic and random components [3]. The flow of a liquid or a homogeneous gas in a pipe was considered. A technique for deriving formulas for the critical Reynolds numbers and the transition point from deterministic to turbulent motion was obtained.

An article by Dmitrenko A.V. [4] deals with a nonisothermal flow in a circular tube in a turbulent regime. The principle of equivalence of measures between laminar and turbulent motions was used. Formulas for the first and second critical Reynolds numbers were obtained.

The calculation of pressure fluctuations in a two-phase turbulent flow was carried out [5]. Solid fine particles in a gas flow were considered. It was found that solid particles strongly affect the turbulence of the gas phase.

Using the stochastic method, Dmitrenko A.V. [6] introduced an uncertainty relation for a family of vortices that describe turbulent phenomena in a pipe and on a flat plate.

The Reynolds analogy between friction and heat transfer in a pipe and on a plate in turbulent flow was studied using the solution of stochastic equations [7].

Recently, it became possible to describe the initial phase of turbulence by terms of the second order of smallness of the Euler continuity equation. The reasons for the appearance of terms of a high order of smallness in the continuity equation are considered below.

Euler gave a lecture “Principles of the Motion of Fluids” in 1752 and in 1755. The French text of Euler’s 1755 lecture “Principes généraux du mouvement des fluides” at the Prussian Royal Academy of Sciences is well-known. It contains an indication that a detailed derivation of the continuity equation can be found in the Latin text of a 1752 report “Principia motus fluidorum” at the Prussian Academy of Sciences. A text of the lecture in French was published in Berlin in 1757 and became widely known. Another text of the report in Latin was published in St. Petersburg four years later, in 1761. The popularity of the Latin text is less than that of the French one.

The differential equation of continuity for an incompressible fluid was derived by L. Euler in 1752 and N.E. Zhukovsky in 1876 by precise geometric calculations.

$$\operatorname{div} \mathbf{V} = 0$$

Here \mathbf{V} is the velocity vector.

In his dissertation “Kinematics of a liquid body” in 1876, Zhukovsky constructed an ellipsoid of deformation and obtained an equation of continuity. Its conclusion can be read in the book [8]. N.E. Zhukovsky defended his master’s thesis at Moscow University. He is known as the author of the formula for the lift of an aircraft wing.

Upon careful reading of Zhukovsky’s master’s thesis in 1997, Professor Bubnov discovered terms of the second order of smallness in the continuity equation [9,10]. Zhukovsky calculated them, but did not use the amount of substance in the balance.

V.A. Bubnov wrote about the presence of terms of the second order of smallness in the continuity equation in a book [2] published in New York in 1998. This is a great merit of the professor of the Moscow City Pedagogical University V.A. Bubnov. In subsequent years, N.E. Zhukovsky did not return to the derivation of the continuity equation.

The author of this review repeated the geometric constructions of Zhukovsky and obtained in 2006 an equation of continuity [11] with terms of a high order of smallness for an incompressible liquid and for a compressible gas. In September 2006, the author presented a report on this finding at the 14th school seminar “Modern problems of aerohydrodynamics” at Moscow State University [12] in Sochi.

Six months later, the author found in Euler’s report from 1752—“Principia motus fluidorum”—an equation of continuity with high-order terms for an incompressible fluid. A comparison [13] showed the identity of the equations of Euler and the author. Euler did not derive an equation of continuity with terms of high order of smallness for a compressible medium.

High-order terms violate the principle of conservation of the amount of matter for an incompressible fluid. Therefore, Euler eliminated these terms by passing to the limit. In a compressible medium, there is a possibility of formation of density waves and pressure waves. The research shows that high-order terms of smallness generate waves. In this case, the conservation principle is not violated.

When developing stochastic methods, it is useful to know about additional high-order terms of the continuity equation. Stochastic equations can describe turbulent flows. However, the initial stage of turbulence can be described by deterministic terms of the second order of smallness of the continuity equation.

Stochastic equations can describe the motion of living matter in biomechanics, but the terms of the second order of smallness of the continuity equation give a description of harmonic self-oscillations too.

The use of high-order terms of smallness of the continuity equation makes it possible to transfer the description of some part of the motions from the stochastic part of the

equation to the deterministic part, which was obtained by Euler by an exact geometric calculation.

2. Equation of Continuity with Terms of High Order of Smallness

2.1. Relationship between Stochastic Description of Turbulence and High-Order Terms of the Continuity Equation

Euler made six deformations of the control figure when deriving the continuity equation. He used three tensile strains and three shear strains along the Cartesian coordinate axes. Six rates of deformation are reflected in the law of conservation of matter. Such deformations transform the cube into a beveled parallelepiped. Its volume gives the dot-vector product. It contains many terms.

$$\begin{aligned} & \left| \begin{array}{ccc} 1 + u'_x \Delta t & u'_y \Delta t & u'_z \Delta t \\ v'_x \Delta t & 1 + v'_y \Delta t & v'_z \Delta t \\ w'_x \Delta t & w'_y \Delta t & 1 + w'_z \Delta t \end{array} \right| = \\ & = 1 + u'_x \Delta t + v'_y \Delta t + u'_x v'_y (\Delta t)^2 + w'_z \Delta t \\ & + u'_x w'_z (\Delta t)^2 + v'_y w'_z (\Delta t)^2 + u'_x v'_y w'_z (\Delta t)^3 + u'_y v'_z w'_x (\Delta t)^3 \\ & + u'_z v'_x w'_z (\Delta t)^3 - u'_z w'_x (\Delta t)^2 - u'_z v'_y w'_x (\Delta t)^3 - u'_y v'_x (\Delta t)^2 \\ & - u'_y v'_x w'_z (\Delta t)^3 - v'_z w'_y (\Delta t)^2 - u'_x v'_z w'_y (\Delta t)^3 \\ & = 1 + \Delta t (u'_x + v'_y + w'_z) \\ & + (\Delta t)^2 \left[\frac{\partial(u,v)}{\partial(x,y)} + \frac{\partial(v,w)}{\partial(y,z)} + \frac{\partial(w,u)}{\partial(z,x)} \right] + (\Delta t)^3 \frac{\partial(u,v,w)}{\partial(x,y,z)} \end{aligned}$$

Here u, v, w are the velocity components along the axes x, y, z ; $t - t_0$ is the control figure deformation time interval; $\frac{\partial(u,v)}{\partial(x,y)}$ and $\partial(u, v, w) / \partial(x, y, z)$ are the Jacobians of the velocity field of the second and third orders, respectively. Euler's continuity equation contains 15 terms.

The last stage of Euler's derivation of the continuity equation and the stochastic method of accounting for turbulent pulsations go in opposite directions. Euler eliminates the higher-order terms of the smallness of the balance of matter by limiting transitions. The stochastic method adds stochastic terms to account for turbulent phenomena. Waves in an incompressible fluid are impossible. Therefore, Euler's position from 1752 is correct.

In 2006, an equation of continuity with terms of high order of smallness for a compressible gas was obtained. The appearance of periodic pressure waves in the gas is possible. The initial stage of harmonic turbulent pulsations can be described by additional terms of the continuity equation with second-order Jacobians. Replacing stochastic terms with the deterministic terms of Euler's geometric calculation is a progressive step.

In Section 2.7, the calculation of the initial stage of turbulization is demonstrated for a flow around a cylinder.

If we made six deformations when deriving the continuity equation, then we should be able to use six parameters in describing the flow. However, six strain rates are combined into three complexes. This is the divergence of the velocity vector, the sum of the second-order Jacobians and the third-order Jacobian. The Jacobians include all six strain rates.

In the divergence of the velocity vector, the rates of shear deformations and the rates of pair deformations in perpendicular directions are not taken into account.

2.2. Derivation of the Continuity Equation for an Incompressible Fluid, Taking into Account the Terms of a High Order of Smallness

Euler's differential continuity equation for an incompressible fluid $div \mathbf{V} = 0$ is an important relation for mathematical physics. Here \mathbf{V} is the velocity vector. This equation reflects the law of conservation of the amount of matter. This equation gives physical

meaning to the mathematical operator div, which was introduced into fluid dynamics. Now it penetrates into various branches of mathematics, physics, and other natural sciences.

D'Alembert's book "Reasoning about the common cause of the winds" from 1747 preceded the derivation of the differential Euler's equation of continuity. D'Alembert calculated the tides in the Earth's atmosphere caused by the Moon. He used the idea of conservation of the amount of matter, but did not derive a differential equation for it.

Euler published the differential equation of continuity in a report "Principia motus fluidorum" at the Prussian Royal Academy of Sciences on 31 August 1752. The talk was in Latin. Euler made a calculation of the change in the shape of the control figure under tension and shear deformations along three Cartesian coordinates with a linear Lagrangian law of deformation in time.

The Cauchy-Helmholtz formulas use the linear dependence of the deformation of the control figure on time and on the initial coordinate of the point

$$x = (1 + at)x_b + bty_b$$

$$y = ctx_b + (1 + kt)y_b$$

where x_b, y_b are initial values of coordinates x, y ; a, b, c, k are constant coefficients; t is time. Important Cauchy-Helmholtz formulas are contained in Section 1 of the book [8]. Linear deformation gives a cubic change in volume over time. Members of the second and third order of smallness violate conservation.

Euler obtained the continuity equation for an incompressible fluid with a large number of terms in 1752. In 1954, C. Truesdell [14] translated the Latin text into English and combined the terms of the higher order of smallness into the Jacobians of the second and third orders.

Euler's continuity equation for an incompressible fluid took the form

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} + (t - t_0) \left[\frac{\partial(u,v)}{\partial(x,y)} + \frac{\partial(v,w)}{\partial(y,z)} + \frac{\partial(w,u)}{\partial(z,x)} \right] + (t - t_0)^2 \partial(u, v, w) / \partial(x, y, z) = 0$$

where u, v, w are the velocity components along the axes x, y, z ; $t - t_0$ is the control figure deformation time interval; $\frac{\partial(u,v)}{\partial(x,y)}$ and $\partial(u, v, w) / \partial(x, y, z)$ are the Jacobians of the velocity field of the second and third orders, respectively.

The terms of the second and third orders of smallness violate the conservation principle for an incompressible fluid. Euler eliminated these terms by passing to the limit $t - t_0 \rightarrow 0$. We see that the higher-order terms of smallness are an intermediate result of inference, but all intermediate results have a physical meaning. Therefore, we have reason to use terms of a high order of smallness in calculating the motion of liquid and gas.

In 1755, Euler gave another lecture at the Prussian Royal Academy of Sciences in French; "Principes généraux du mouvement des fluides". The French text did not contain high-order terms in the continuity equation.

2.3. Derivation of the Continuity Equation for a Compressible Liquid and Gas, Taking into Account the Terms of a High Order of Smallness

In 1876, N.E. Zhukovsky made the construction of a deformation ellipsoid in his master's thesis and obtained the equation of continuity. Professor V.A. Bubnov noticed in 1997 that Zhukovsky calculated some terms of the second order of smallness [2,9,10].

In 2006, the author repeated the conclusion of N.E. Zhukovsky and obtained an equation of continuity with terms of the second and third orders of smallness [1,11,12,15]. It was concluded for an incompressible liquid and for a compressible gas.

For a compressible gas, the continuity equation has the form

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} + (t - t_0)\rho \left[\frac{\partial(u,v)}{\partial(x,y)} + \frac{\partial(v,w)}{\partial(y,z)} + \frac{\partial(w,u)}{\partial(z,x)} \right] + (t - t_0)^2 \rho \partial(u, v, w) / \partial(x, y, z) = 0$$

where ρ is the density.

After some time, the author found the continuity equation for an incompressible fluid in Euler’s 1752 work “Principia motus fluidorum”. A comparison confirmed the identity of the author’s and Euler’s equations of continuity for an incompressible fluid. Euler did not write out an equation of continuity with terms of a high order of smallness for a compressible gas. The 2006 continuity equation for a compressible gas is new.

2.4. Derivation of the Wave Equation

The homogeneous wave equation is the d’Alembert operator of pressure, which is equal to zero. A homogeneous wave equation can describe the transfer of harmonic vibrations that are generated by boundary conditions. The internal volume of the gas does not generate pressure waves and density waves. The inhomogeneous part of the inhomogeneous wave equation generates harmonic waves within the gas flow.

Lighthill added a turbulent term to the continuity equation from 1952–1954. His method of acoustic analogy gives an inhomogeneous wave equation with a turbulent term in the inhomogeneous part. This heterogeneous term generates turbulent pulsations.

The terms of a high order of smallness of the continuity equation behave in a similar way. Of these, the Jacobians of the second order of smallness fall into the inhomogeneous part of the wave equation and generate harmonic waves.

The physical meaning of additional terms of the second order of smallness was understood after the derivation of the inhomogeneous wave equation [13]. In 2007, the author derived an inhomogeneous wave equation using the acoustic analogy method [16,17] of Lighthill.

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} - \left(\frac{1}{c_0^2} \right) \frac{\partial^2 p}{\partial t^2} = \rho_0 \left[\frac{\partial(u,v)}{\partial(x,y)} + \frac{\partial(v,w)}{\partial(y,z)} + \frac{\partial(w,u)}{\partial(z,x)} \right] + (t - t_0)\rho_0 2 \frac{\partial(u,v,w)}{\partial(x,y,z)}$$

Here c_0 is the speed of sound, ρ_0 is thermodynamic density.

The sum of the second-order Jacobians goes over to the right inhomogeneous side of the wave equation. We can see that the square of the sum of the Jacobians controls the intensity of the sound generation [13,18,19].

The wave equation

$$\frac{c_0^{-2} \partial^2 p}{\partial t^2} - \left(\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} \right) = -\rho_0 J \tag{1}$$

$$J = \frac{\partial(u, v)}{\partial(x, y)} + \frac{\partial(v, w)}{\partial(y, w)} + \frac{\partial(w, u)}{\partial(z, x)}$$

has a solution of the lagging potential type for the sound pressure

$$p(\mathbf{r}, t) = \frac{\rho_0}{4\pi} \int J \Big|_{t-R/c} R^{-1} dW$$

Here $R = |\mathbf{r} - \mathbf{r}_1|$, \mathbf{r} is the radius vector of the observation point, \mathbf{r}_1 is the radius vector of the point in the region of integration.

The intensity of generation of periodic oscillations can be estimated by the formula

$$I \left[\frac{W}{m^2} \right] = \frac{p^2}{c_0 \rho_0} = \frac{\rho_0 J^2 W^2}{16\pi^2 c_0 r^2} \tag{2}$$

if the difference between the sum of Jacobians J from zero is observed only in the volume W .

The wave equation describes the sound and self-oscillations that arise in the flow when flowing around solids. This solution can be used to calculate harmonic vibrations due to wind in bridges and long structures. The solution to the differential inhomogeneous wave equation has the form of harmonic oscillations. They lead to vibration of a rigid body that flows around the stream.

Turbulent flow contains disordered pressure pulsations and pulsations of velocity components. However, the initial phase of turbulence has harmonic oscillations, which are superimposed on the stationary flow.

A large number of velocity fields are known for potential currents. For them, we can calculate the intensity of the vibrations that arise. We will use a stationary potential velocity field in the right inhomogeneous side of the wave equation. The left side of the wave equation contains the derivatives of the acoustic wave pressure. The method of successive approximations is used.

This is the known flat potential flow inside the right angle for the complex potential $w = z^2$.

$$w = \varphi + i\psi$$

Here $z = x + iy$ is a complex argument, φ is velocity potential, ψ is stream function. Velocity components u , v along the coordinate axes x, y are given by

$$u = \frac{\partial \varphi}{\partial x} = \frac{\partial(x^2 - y^2)}{\partial x} = 2x$$

$$v = \frac{\partial \varphi}{\partial y} = \frac{\partial(x^2 - y^2)}{\partial y} = -2y$$

and Jacobian is constant within the flow region.

We can use solution (2) to estimate the intensity of sound generation in other flows for which the velocity field is known. The terms of the second and third orders of smallness of the continuity equation for an incompressible fluid violate the law of conservation of matter. In the case of plane two-dimensional motion without shear deformations, the area of the control figure decreases according to the parabolic law.

This reduction in area is called "the shagreen leather effect" by analogy with Honoré Balzac's novel.

2.5. Differential (Local) and Integral Conservation

The use of terms of the second order of smallness in a differential equation raises the question of the possibility of differential or local nonconservation [20]. The conservation law was formulated in antiquity in many philosophical teachings of the world. It came to Europe from the school of Pythagoras (570–490 BC). Empedocles (490–430 BC) gave it a written form. The ancients attributed the principle of conservation to one large isolated object. This is integral conservation.

In the differential equation of continuity, we are talking about the conservation or nonconservation of the amount of matter in small local or differential volumes. There are many such volumes. Each test volume has 14 contiguous volumes. The substance from the control volume can flow for some time into adjacent volumes. After a while, this substance can return to the control volume. This process is described by additional terms of the continuity equation.

The addition of terms to the continuity equation first occurred in 1951. The continuity equation with allowance for diffusion was proposed by S.V. Wallander [21] and N.A. Slezkin [22]. S.V. Wallander was the vice-rector of Leningrad State University. N.A. Slezkin was the dean of the Faculty of Mechanics and Mathematics at Moscow State University. The authors have not proposed technical or physical problems to use this equation. The equation has not been used.

Euler’s equation of continuity with local nonconservation describes the formation of sound in a wind musical instrument. Local nonconservation of the amount of air occurs inside the flute. The A note of the first octave has a frequency of 440 Hz. For this note, we get 880 times per second the conservation of the amount of substance inside the flute. Local nonconservation is the reason for the flute’s sound generation. Nonconservation is the stimulus for the formation of density waves. This is Euler’s great find.

It cannot be hidden and covered up. It cannot be retouched. It must be studied and used. The wave Equation (1) allows you to calculate the intensity of the generation of sound by a flow of liquid or gas.

The use of the exponential Lagrangian law of motion of a liquid particle [23] in the derivation of the continuity equation does not provide terms of a high order of smallness in the continuity equation. The change in the coordinates of the points of the fluid is found from the solution of the Cauchy problem for a system of differential equations

$$\frac{dx}{dt} = ax + by$$

$$\frac{dy}{dt} = cx + ky$$

with initial conditions $x = x_b, y = y_b$ at $t = 0$. Here a, b, c, k are constant coefficients.

The classical equation of continuity is obtained without additional terms of a high order of smallness. The analysis shows that the exponential law of motion of a liquid particle corresponds to an accelerated flow of a liquid.

Turbulent pulsations and oscillations do not occur in the region of accelerated flow. This is confirmed by experimental studies. The continuity equation for the region of accelerated flow does not contain terms of a high order of smallness, but the flow field cannot consist only of the region of the accelerated flow. Acoustic waves are generated in areas without acceleration.

Euler used the linear law of motion of a liquid particle to derive the continuity equation. The linear law is fulfilled in areas of flow without acceleration. These areas generate self-oscillation and sound.

The same high-order terms of smallness are found in another derivation of the continuity equation which uses the Gauss-Ostrogradsky divergence theorem.

2.6. High-Order Terms in the Derivation of the Continuity Equation Using the Gauss-Ostrogradsky Theorem

M.V. Ostrogradsky’s derivation of the continuity equation is based on the famous theorem which was published in his article “Note sur la theorie de la chaleur”. Geometric constructions of the Gauss-Ostrogradsky theorem give high-order terms in the continuity equation, but high-order terms disappear when using the transitions to the limits in the integrals $\Delta x \rightarrow 0, \Delta y \rightarrow 0, \Delta z \rightarrow 0$. The geometric constructions of the Gauss-Ostrogradsky theorem calculate the balance of the fluid that flows into the control figure over time $\Delta t = t - t_0$ and flows out of the figure over time $\Delta t = t - t_0$. The theorem takes into account liquid particles that have crossed the boundary once per time $t - t_0$.

There are streamlines that follow the secant line near the boundary of the convex control figure. A liquid particle can cross the border of the control two times [23] in time interval $t - t_0$.

Such liquid particles leave the convex control figure (Figure 1). This is a property of convex figures. These particles reduce the balance of matter, which is controlled by the Gauss-Ostrogradsky theorem

$$\iiint_W \left(\frac{dP}{dx} + \frac{dQ}{dy} + \frac{dR}{dz} \right) dx dy dz = \iint_S (P \cos \lambda + Q \cos \mu + R \cos \nu) dS$$

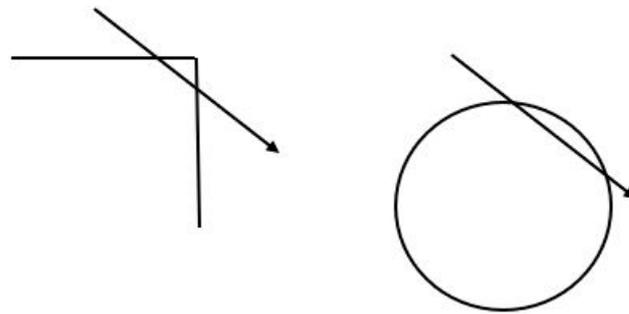


Figure 1. The secants along which the liquid particle crosses the boundary two times in the time interval $t - t_0$.

The Gauss-Ostrogradsky theorem uses direction cosines [23] and ignores such streamlines. Allowance for double crossing of the boundary in time $t - t_0$ gives terms of a high order of smallness.

The integral is the result of going to the limit in the integral sum. The geometric formulation of the conservation principle is done for integral sums. Integrals are secondary concepts and integral sums are primary concepts. We must give priority to integral sums. We can calculate the amount of liquid that crosses the boundary two times in time $t - t_0$. This amount gives terms of the second order of smallness in time. Calculation of the flow around a right angle gives the same value of the terms of the second order of smallness in the Euler continuity equation and in the Gauss-Ostrogradsky theorem, in which the integrals are replaced by integral sums.

Similar terms of higher order of smallness were denoted by ϵ in the section “The Gauss-Ostrogradsky Theorem” in a textbook by academician Sedov L.I.—head of the Department of Hydromechanics at Moscow State University [24]. For ϵ , you can write an expression in the form of a formula.

The Gauss-Ostrogradsky formula for plane flow

$$\iint_S \left(\frac{dP}{dx} + \frac{dQ}{dy} \right) dx dy = \int_L (P \cos \lambda + Q \cos \mu) dL$$

in integral sums using terms of the second order of smallness takes the form

$$\Sigma \Sigma_S \left[\partial P / \partial x + \partial Q / \partial y + (t - t_0) \begin{bmatrix} \partial P / \partial x & \partial P / \partial y \\ \partial Q / \partial x & \partial Q / \partial y \end{bmatrix} \right] \Delta x \Delta y = \Sigma_L (P \Delta y + Q \Delta x)$$

Second-order terms of smallness are important for use in a wave differential equation that has second-order derivatives. The terms of the second order of smallness disappear when using the transitions to the limits in the integrals. Limit transitions reduce information about fluid motion.

We have confirmation of this position.

The Gauss-Ostrogradsky theorem for integral sums is discussed in the article [25].

2.7. Generation of Periodic Waves When Flowing Around a Cylinder

We will demonstrate the use of the inhomogeneous wave equation derived in Section 2.4. The generation of waves when a gas flow around a cylinder is described below.

The wave equation is given by

$$\left(\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} + \frac{\partial^2 p}{\partial z^2} \right) - \frac{c_0^{-2} \partial^2 p}{\partial t^2} = \rho_0 \left(\frac{\partial(u, v)}{\partial(x, y)} + \frac{\partial(v, w)}{\partial(y, w)} + \frac{\partial(w, u)}{\partial(z, x)} \right)$$

where p is the acoustic pressure. The inhomogeneous part of the wave equation contains the second-order Jacobian, which comes from the additional terms of the Euler continuity equation.

The second-order Jacobians in the inhomogeneous right-hand side of the equation generate periodic harmonic waves. The complex potential $w = z + 1/z$ makes it possible to calculate the potential stationary transverse flow around the cylinder. The velocity field is given by the following formulas

$$u = \frac{\partial \varphi}{\partial x} = 1 + \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

$$v = \frac{\partial \varphi}{\partial y} = \frac{-2xy}{(x^2 + y^2)^2}$$

The calculation of the Jacobian J gives the following result

$$J = -4 \cos(6\varphi) / r^6$$

where φ is the polar angle, r is radius.

Dependence of the intensity of generation of oscillations on the polar angle φ and radius r is given by

$$I \sim \frac{\cos^2(6\varphi)}{r^{12}} \sim [1 + \cos(12\varphi)] / (2r^{12})$$

We got a standing wave (Figure 2). The standing wave has 12 antinodes every 30° around the circumference. The standing wave has 12 knots every 30° around the circumference. In the antinodes, intensification of diffusion and heat exchange of the cylinder with the environment can occur.

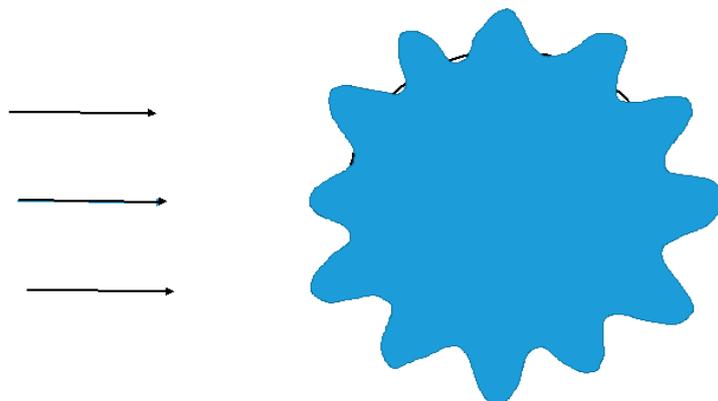


Figure 2. A system of standing waves with a potential flow around a cylinder.

2.8. The Use of Terms of the Second Order of Smallness in the Regularization of the Continuity Equation

The solution of the equations of hydrodynamics and gas dynamics by numerical methods on a computer has the problem of slow convergence of iterations. An important way to improve convergence is to add terms of the second order of smallness to the continuity equation and to the equation of motion. This is a method for regularizing equations. Elizarova T.G. and her students [26,27] use terms of the second order of smallness to regularize the continuity equation, which coincide with Euler’s additional terms [28]. The use of additional second-order terms with a small coefficient led to the construction of effective numerical algorithms for solving the equations of hydrodynamics and gas dynamics. The corresponding algorithms were included in the open source software package OpenFOAM in the form of new computing cores [27]. These algorithms are now available for widespread use.

The use of these additives makes it possible to expand the range of obtaining numerical solutions for the flow parameters.

Walter Pauls [29] translated Euler’s 1752 report into English.

My group translated from Latin into many languages only the initial sections of this lecture, which contain the derivation of the equation of continuity [30–32]. The translation was done by V.M. Ovsyannikov, E.V. Ivanova, and A.A. Ovsyannikova.

It is now possible to teach at universities the Euler equation of continuity with terms of high order of smallness, as an additional section of the mechanics of liquid and gas.

Authors of stochastic additives of a similar purpose should be aware of Euler's exact results and be able to use them.

3. Conclusions

1. A more complete equation of continuity was found for an incompressible fluid with terms of a high order of smallness in a hard-to-reach place. A more complete equation of continuity was found within the Latin text of Euler's classical derivation of 1752.
2. The presence of similar additional terms of high order of smallness was found in the constructions of the integral sums of the Gauss-Ostrogradsky divergence theorem of 1831.
3. In 2006, a complete equation of continuity with terms of high order of smallness for a compressible gas was proposed which fulfils the principle of conservation of the amount of matter.
4. A new concept of local nonconservation proposed by the Keldysh Institute of Applied Mathematics was described.
5. An opportunity was obtained to count the harmonic oscillations that arise against the background of a stationary flow.
6. An example of calculating standing waves for a transverse flow around a cylinder was given. The calculation was made using the complete equation of continuity with terms of the second order of smallness. The result is in the form of a "virus". This is an illustration of a new class of problems that can be solved using the complete continuity equation with terms of the second order of smallness.
7. For many years, additional terms of the continuity equation have been used in the method of regularization of hydrodynamic equations to improve the convergence of numerical simulation iterations. In 2017, it was found that the additional terms of the regularization method agree with the terms derived by Euler.
8. Stochastic equations can describe the initial stage of turbulization—the movement of living matter. The use of high-order terms of smallness of the Euler continuity equation makes it possible to transfer the description of some part of the motions from the stochastic part of the equation to the deterministic part.
9. It is now possible to teach the Euler equation of continuity with terms of high order of smallness at universities as an additional section of the mechanics of liquid and gas.

Funding: This research received no external funding.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Ovsyannikov, V.M. N.E. Zhukovsky's strain ellipsoid taking into account terms of the second order of smallness. *Eng. J Sci. Innov.* **2018**, *77*, 1–11. Available online: <http://www.engjournal.ru/articles/1756/1756.pdf> (accessed on 16 April 2021). (In Russian) [[CrossRef](#)]
2. Bubnov, V.A. *Convective Heat and Mass Transfer in an Insulated Trailing Swirl*; Begell House Publishers: New York, NY, USA, 1998; p. 174.
3. Dmitrenko, A.V. Equivalence of Measures and Stochastic Equations for Turbulent Flows. *Dokl. Phys.* **2013**, *58*, 228–235. [[CrossRef](#)]
4. Dmitrenko, A.V. Analytical estimation of velocity and temperature fields in a circular tube on the basis of stochastic equations and equivalence of measures. *J. Eng. Phys. Thermophys.* **2015**, *88*, 1569–1576. [[CrossRef](#)]
5. Dmitrenko, A.V. Calculation of pressure pulsations for a turbulent heterogeneous medium. *Dokl. Phys.* **2007**, *52*, 384–387. [[CrossRef](#)]
6. Dmitrenko, A.V. The uncertainty relation in the turbulent continuous medium. *Contin. Mech. Thermodyn.* **2020**, *32*, 161–171. [[CrossRef](#)]

7. Dmitrenko, A.V. Reynolds analogy based on the theory of stochastic equations and equivalence of measures. *Eng. Phys. J.* **2021**, *94*, 195–202. (In Russian)
8. Kochin, N.E.; Kibel, I.A.; Rose, N.V. *Theoretical hydromechanics. Part I*; OGIZ: Moscow, Russia, 1941; p. 348. (In Russian)
9. Bubnov, V.A. Physical principles of hydrodynamic motion. In *Axiomatical Problems in Hydro-Gasdynamics. № 4*; PROMETHEY: Moscow, Russia, 1997; pp. 206–269. ISBN 5-7042-0851-7. (In Russian)
10. Bubnov, V.A. Kinematics of a liquid particle. In *Axiomatical Problems in Hydro-Gasdynamics. № 7*; PROMETHEY: Moscow, Russia, 1999; pp. 11–29. ISBN 5-7042-0938-6. (In Russian)
11. Ovsyannikov, V.M. The introduction to axiomatic Hydrodynamics founded on the basic experiments with the liquid. In *Axiomatical Problems in Hydro-Gasdynamics. № 15*; Wek knigi: Moscow, Russia, 2006; pp. 19–51. ISBN 5-94436-018-X. (In Russian, English, German, French)
12. Ovsyannikov, V.M. Difference form of the equation of continuity of a continuous medium taking into account the bending deformations of the coordinate angles. In Proceedings of the 14th School-Seminar—Modern Problems of Aerohydrodynamics, Sochi, Russia, 6–16 September 2006; p. 74. (In Russian)
13. Ovsyannikov, V.M. The occurrence of non-stationary flow under the stationary initial and boundary conditions. In *Axiomatical Problems in Hydro-Gasdynamics. № 16*; Sputnik+: Moscow, Russia, 2007; pp. 13–38. ISBN 978-5-364-00601-1. (In Russian, English, German, French)
14. Euler, L. *Commentationes Mechanicae ad Theoriam Corporum Pertinentes. Volumen Prius*; Truesdell, C.A., Ed.; Societas Scientiarum Naturalium Helveticae: Lausannae, Stwitzerland, 1954.
15. Ovsyannikov, V.M.; Using, L. Euler’s equation of continuity in the environment of unstable current. *Vestn. Mosc. City Teach. Train. Univ. Ser. Nat. Sci.* **2010**, *6*, 21–25. (In Russian)
16. Lighthill, M.J. On sound generated aerodynamically. Part I: General theory. *Proc. R. Soc. Lond.* **1952**, *A221*, 564–587.
17. Lighthill, M.J. On sound generated aerodynamically. Part II: Turbulence as a source of sound. *Proc. R. Soc. Lond.* **1954**, *A222*, 1–32.
18. Ovsyannikov, V.M. Identity of the right part of Phillips’s wave equation of sound generation with an additional member of Euler’s finite difference continuity equation. *Vestn. Mosc. City Teach. Train. Univ. Ser. Nat. Sci.* **2011**, *8*, 16–21. (In Russian)
19. Ovsyannikov, V.M. Landau—Lifshits vibrator in the equations of gas dynamics. *Eng. J. Sci. Innov.* **2018**, *76*, 1–8. Available online: <http://www.engjournal.ru/articles/1739/1739.pdf> (accessed on 16 April 2021). (In Russian) [CrossRef]
20. Ovsyannikov, V.M. Local non-conservation at periodic oscillatory movements. *Vestn. Mosc. City Teach. Train. Univ. Ser. Nat. Sci.* **2015**, *17*, 25–29. Available online: https://www.mgpu.ru/uploads/adv_documents/2526/1484827749-VestnikEstNauki172015.Pdf (accessed on 16 April 2021). (In Russian)
21. Wallander, S.V. Equations of motion of a viscous gas. *Rep. Acad. Sci. USSR* **1951**, *78*, 25–27. (In Russian)
22. Slezkin, N.A. On differential equations of gas motion. *Rep. Acad. Sci. USSR* **1951**, *77*, 205–208. (In Russian)
23. Ovsyannikov, V.M. Using time-linear Cauchy—Helmholtz formulas in the derivations of the continuity equation of Euler, Ostrogradsky, Zhukovsky. *Eng. J. Sci. Innov.* **2020**, *101*, 1–15. Available online: <http://www.engjournal.ru/articles/1978/1978.pdf> (accessed on 16 April 2021). (In Russian) [CrossRef]
24. Sedov, L.I. *Mechanics of Continuous Media*; Series in Theoretical and Applied Mechanics; World Scientific: Singapore, 1997; Volume 2, p. 1368.
25. Ovsyannikov, V.M. Euler’s continuity equation with high-order terms of smallness in flow time. *Results Sci. Technol. Ser. Contemp. Math. Appl. Themat. Rev.* **2020**, *182*, 95–100. (In Russian)
26. Elizarova, T.G. *Quasi-Gas Dynamic Equations*; Springer: Berlin, Germany, 2009.
27. Kraposhin, M.V.; Smirnova, E.V.; Elizarova, T.G.; Istomina, M.A. Development of a new OpenFOAM solver using regularized gas dynamic equations. *Comput. Fluids* **2018**, *166*, 163–175. [CrossRef]
28. Ovsyannikov, V.M. Comparison of additional second-order terms in finite-difference Euler equations and regularized fluid dynamics equations. *J. Comput. Math. Math. Phys.* **2017**, *57*, 876–880. [CrossRef]
29. Euler, L. Principia motus fluidorum. *arXiv* **2008**, (Translated by Pauls, L.W.). arXiv:0804.4802v1.
30. Euler, L. *Principles of the Motion of Fluids*; Ivanova, E.V.; Ovsyannikov, V.M., Translators; Sputnik+: Moscow, Russia, 2018; p. 93.
31. Euler, L. *Principles of the Motion of Fluids*, 2nd enlarged ed.; Ivanova, E.V.; Ovsyannikov, V.M., Translators; Sputnik+: Moscow, Russia, 2019; p. 177.
32. Euler, L. *Principles of the Motion of Fluids*, 4th updates ed.; Ivanova, E.V.; Ovsyannikov, V.M., Translators; Sputnik+: Moscow, Russia, 2020; p. 203.