fractal and fractional

MDPI

Article

Generalized Integral Inequalities of Chebyshev Type

Paulo M. Guzmán [1,2,3], Péter Kórus [4] and Juan E. Nápoles Valdés [1,5,*]

1 Facultad de Ciencias Exactas y Naturales y Agrimensura, Universidad Nacional del Nordeste, Corrientes 3400, Argentina; paulomatiasguzman@hotmail.com
2 Facultad de Ingeniería, Universidad Nacional del Nordeste, Resistencia, Chaco 3500, Argentina
3 Facultad de Ciencias Agrarias, Universidad Nacional del Nordeste, Corrientes 3400, Argentina
4 Department of Mathematics, Juhász Gyula Faculty of Education, University of Szeged, Hattyas utca 10, H-6725 Szeged, Hungary; korpet@jgypk.szte.hu
5 Facultad Regional Resistencia, Universidad Tecnológica Nacional, French 414, Resistencia, Chaco 3500, Argentina
* Correspondence: jnapoles@exa.unne.edu.ar

Received: 29 February 2020; Accepted: 27 March 2020; Published: 2 April 2020

check for updates

Abstract: In this paper, we present a number of Chebyshev type inequalities involving generalized integral operators, essentially motivated by the earlier works and their applications in diverse research subjects.

Keywords: Chebyshev inequality; integral inequality; fractional calculus

MSC: 26D10; 26A33

---

1. Introduction

Many integral inequalities of various types have been presented in the literature. Among them, we choose to recall the following Chebyshev inequality (see [1]):

$$\frac{1}{b-a}\int_a^b f(x)g(x)dx \geq \left(\frac{1}{b-a}\int_a^b f(x)dx\right)\left(\frac{1}{b-a}\int_a^b g(x)dx\right), \tag{1}$$

where f and g are two integrable and synchronous functions on $[a,b]$, $a<b$, $a,b\in\mathbb{R}$. Here, two functions f and g are called *synchronous* on $[a,b]$ if

$$(f(x)-f(y))(g(x)-g(y)) \geq 0 \quad (x,y\in[a,b]).$$

In the case that we have $-f$ and g (or similarly f and $-g$) the sense of the previous inequality is the opposite.

Inequality (1) has many applications in diverse research subjects such as numerical quadrature, transform theory, probability, existence of solutions of differential equations and statistical problems. Many authors have investigated generalizations of the Chebyshev inequality (1), these are called Chebyshev type inequalities (see, e.g., [2,3] or [4]).

We give the definition of a general fractional integral. We assume that the reader is familiar with the classic definition of the Riemann integral, so we will not present it. Throughout the paper we will suppose that the positive integral operator kernel $T: I \to (0,\infty)$ defined below is an absolutely continuous function on interval $I\subseteq\mathbb{R}$.

Definition 1. *Let I be an interval $I\subseteq\mathbb{R}$ and $a,b\in I$. The generalized integral operators J_{T,a^+} and J_{T,b^-}, called respectively, right and left, are defined for every locally integrable function f on I as follows:*

$$J_{T,a^+}(f)(x) = \int_a^x \frac{f(t)}{T(t-a)}dt, \quad x > a.$$

$$J_{T,b^-}(f)(x) = \int_x^b \frac{f(t)}{T(b-t)}dt, \quad x < b.$$

Note that in special cases, J_{T,a^+} and J_{T,b^-} are equal to the following integrals:

$$J_{T,0^+}(f)(1) = \int_0^1 \frac{f(t)}{T(t)}dt$$

and

$$J_{T,1^-}(f)(0) = \int_0^1 \frac{f(t)}{T(1-t)}dt = \int_0^1 \frac{f(1-t)}{T(t)}dt.$$

We say that f belongs to the function space $L_T^+[a,b]$ if

$$J_{T,a^+}(f)(b) < \infty,$$

similarly f belongs to $L_T^-[a,b]$ if

$$J_{T,b^-}(f)(a) < \infty,$$

and $f \in L_T[a,b]$ if $f \in L_T^+[a,b] \cap L_T^-[a,b]$.

It is easy to see that the case of the J_T operators defined above contains, as particular cases, the integral operators obtained from conformable and non-conformable local derivatives. For details about the Riemann–Liouville fractional integrals (left-sided) of a function f of order $\alpha \in \mathbb{C}$ with $\mathrm{Re}(\alpha) > 0$ the reader can consult [5,6]. In [7], Belarbi and Dahmani established some theorems related to the Chebyshev inequality involving Riemann–Liouville fractional integral operator. Recently, some new integral inequalities involving this fractional integral operator have appeared in the literature, see, e.g., [8–19].

Taking into account the previous research results and the generalized integral operator, we will obtain some Chebyshev type inequalities, which contain many of the inequalities reported in the literature as particular cases.

2. Main Results

Theorem 1. *Let f and g be two functions from $L_T^+[a,b]$ which are synchronous on $[a,b]$. Then*

$$J_{T,a^+}(fg)(b) \geq [\tau(b-a)]^{-1} J_{T,a^+}(f)(b)\, J_{T,a^+}(g)(b) \tag{2}$$

where

$$\tau(x) = \int_0^x \frac{ds}{T(s)}.$$

Proof. Since f and g are synchronous on $[a,b]$, we have

$$(f(u)-f(v))(g(u)-g(v)) \quad \geq \quad 0; \quad u,v \in [a,b]$$

or equivalently

$$f(u)g(u) + f(v)g(v) \geq f(u)g(v) + f(v)g(u).$$

Multiplying both sides by $\frac{1}{T(u-a)}$ yields

$$\frac{f(u)g(u)}{T(u-a)} + \frac{f(v)g(v)}{T(u-a)} \geq \frac{f(u)g(v)}{T(u-a)} + \frac{f(v)g(u)}{T(u-a)}.$$

Integrating both sides of the resulting inequality with respect to the variable u from a to b, gives us

$$\int_a^b \frac{f(u)g(u)}{T(u-a)}du + \int_a^b \frac{f(v)g(v)}{T(u-a)}du \geq \int_a^b \frac{f(v)g(u)}{T(u-a)}du + \int_a^b \frac{f(u)g(v)}{T(u-a)}du.$$

From this, we have

$$J_{T,a^+}(fg)(b) + f(v)g(v)\tau(b-a) \geq g(v)\, J_{T,a^+}(f)(b) + f(v)\, J_{T,a^+}(g)(b). \tag{3}$$

After multiplying the inequality by $\frac{1}{T(v-a)}$ and integrating with respect to v between a and b, we get

$$\begin{aligned} &J_{T,a^+}(fg)(b)\ \tau(b-a) + \tau(b-a)\int_a^b \frac{f(v)g(v)}{T(v-a)}dv \\ &\geq J_{T,a^+}(f)(b)\int_a^b \frac{g(v)}{T(v-a)}dv + J_{T,a^+}(g)(b)\int_a^b \frac{f(v)}{T(v-a)}dv, \end{aligned}$$

that is

$$2\, J_{T,a^+}(fg)(b)\ \tau(b-a) \geq 2\, J_{T,a^+}(f)(b)\ J_{T,a^+}(g)(b)$$

and we have got (2). □

Remark 1. *Similar calculations as above shows that for any $f, g \in L_T^-[a,b]$ synchronous on $[a,b]$, we have*

$$J_{T,b^-}(fg)(a) \geq [\tau(b-a)]^{-1} J_{T,b^-}(f)(a)\ J_{T,b^-}(g)(a). \tag{4}$$

Remark 2. *If we take $T \equiv 1$ in Theorem 1 (or in Remark 1), then inequality (2) (or (4)) reduces to the classic inequality (1) of Chebyshev.*

Remark 3. *If we consider the kernel ($\alpha, \beta > 0$)*

$$T(x-t) = T(x-t,\alpha,\beta) = \frac{\Gamma(\beta)t^{1-\alpha}}{\left(\frac{\alpha}{x^\alpha - t^\alpha}\right)^{\beta-1}}, \tag{5}$$

we obtain ([16], Theorem 5) that contains ([7], Theorem 3.1) as a particular case.

Theorem 2. *Let f and g be two functions from $L_{T_1}^+[a,b] \cap L_{T_2}^+[a,b]$ which are synchronous on $[a,b]$. Then*

$$\begin{aligned} &\tau_2(b-a)\, J_{T_1,a^+}(fg)(b) + \tau_1(b-a)\, J_{T_2,a^+}(fg)(b) \\ &\geq J_{T_1,a^+}(f)(b)\ J_{T_2,a^+}(g)(b) + J_{T_1,a^+}(g)(b)\ J_{T_2,a^+}(f)(b). \end{aligned} \tag{6}$$

where

$$\tau_1(x) = \int_0^x \frac{ds}{T_1(s)} \quad \text{and} \quad \tau_2(x) = \int_0^x \frac{ds}{T_2(s)}.$$

Proof. Writing T_1 in place of T and τ_1 in place of τ in (3) and then multiplying both sides by $\frac{1}{T_2(v-a)}$ yields

$$\frac{J_{T_1,a^+}(fg)(b)}{T_2(v-a)} + \tau_1(b-a)\frac{f(v)g(v)}{T_2(v-a)} \geq J_{T_1,a^+}(f)(b)\frac{g(v)}{T_2(v-a)} + J_{T_1,a^+}(g)(b)\frac{f(v)}{T_2(v-a)}.$$

Integrating both sides of the resulting inequality with respect to the variable v between a and b gives us (6). □

Remark 4. *In case of $T_1 = T_2$, we obtain Theorem 1.*

Remark 5. *By taking the kernels ($\alpha, \beta, \tau > 0$)*

$$T_1(x-t) = \frac{\Gamma(\beta)t^{1-\alpha}}{\left(\frac{\alpha}{x^\alpha - t^\alpha}\right)^{\beta-1}} \quad \text{and} \quad T_2(x-t) = \frac{\Gamma(\beta)t^{1-\tau}}{\left(\frac{\tau}{x^\tau - t^\tau}\right)^{\beta-1}},$$

we obtain ([16], Theorem 6) and hence ([7], Theorem 3.2) as a particular case.

Theorem 3. *Let $\{f_i\}_{i=1,2,\dots,n}$ be positive increasing functions from $L_T^+[a,b]$. We have*

$$\left[J_{T,a^+}\left(\prod_{i=1}^n f_i\right)(b)\right] \geq [\tau(b-a)]^{1-n}\left[\prod_{i=1}^n J_{T,a^+}(f_i)(b)\right]. \tag{7}$$

Proof. We prove this theorem by induction on $n \in \mathbb{N}$. For $n = 1$, (7) trivially holds. For $n = 2$, (7) immediately comes from (2), since f_1 and f_2 are synchronous on $[a,b]$. Now assume that the inequality (7) is true for some $n \in \mathbb{N}$. Let $f := \prod_{i=1}^n f_i$ and $g := f_{n+1}$. Observe that f and g are increasing on $[a,b]$, therefore (2) and the induction hypothesis for n yields

$$\begin{aligned} J_{T,a^+}\left(\prod_{i=1}^n f_i\, f_{n+1}\right)(b) &\geq [\tau(b-a)]^{-1} J_{T,a^+}\left(\prod_{i=1}^n f_i\right)(b)\, J_{T,a^+}(f_{n+1})(b) \\ &\geq [\tau(b-a)]^{-n}\prod_{i=1}^{n+1} J_{T,a^+}(f_i)(b). \end{aligned}$$

This completes the induction and the proof. □

Remark 6. *Taking kernel (5), we obtain ([16], Theorem 7), which is a generalization of ([7], Theorem 3.3).*

Theorem 4. *Let $f, g : [0,\infty) \to \mathbb{R}$, $f, g \in L_T^+[a,b]$ such that f is increasing and g is differentiable with g' bounded below by $m = \inf_{t\in[0,\infty)} g'(t)$. Then we have*

$$\begin{aligned} J_{T,a^+}(fg)(b) &\geq [\tau(b-a)]^{-1} J_{T,a^+}(f)(b)\, J_{T,a^+}(g)(b) \\ &\quad - \frac{m}{\tau(b-a)} J_{T,a^+}(f)(b)\, J_{T,a^+}(t)(b) + m\, J_{T,a^+}(tf)(b), \end{aligned}$$

where $t(x) = x$ is the identity function.

Proof. Let $p(x) = mx$ and $h(x) = g(x) - p(x)$. Note that h is differentiable and increasing on $[0,\infty)$. Hence we can apply (2), and we obtain

$$\begin{aligned} J_{T,a^+}(fh)(b) &\geq [\tau(b-a)]^{-1} J_{T,a^+}(f)(b)\, J_{T,a^+}(h)(b) \\ &= [\tau(b-a)]^{-1} J_{T,a^+}(f)(b)\, J_{T,a^+}(g)(b) \\ &\quad - [\tau(b-a)]^{-1} J_{T,a^+}(f)(b)\, J_{T,a^+}(p)(b). \end{aligned} \tag{8}$$

Since

$$J_{T,a^+}(p)(b) = m\, J_{T,a^+}(t)(b)$$

and

$$J_{T,a^+}(fp)(b) = m\, J_{T,a^+}(tf)(b),$$

(8) implies

$$\begin{aligned} J_{T,a^+}(fg)(b) &= J_{T,a^+}(fh)(b) + J_{T,a^+}(fp)(b) \\ &\geq [\tau(b-a)]^{-1} J_{T,a^+}(f)(b)\, J_{T,a^+}(g)(b) \\ &\quad - [\tau(b-a)]^{-1} J_{T,a^+}(f)(b)\, J_{T,a^+}(p)(b) + J_{T,a^+}(fp)(b) \\ &\geq [\tau(b-a)]^{-1} J_{T,a^+}(f)(b)\, J_{T,a^+}(g)(b) \\ &\quad - \frac{m}{\tau(b-a)}\, J_{T,a^+}(f)(b)\, J_{T,a^+}(t)(b) + m\, J_{T,a^+}(tf)(b), \end{aligned}$$

where the desired result is obtained. □

Remark 7. *Using kernel* (5)*, we obtain (*[16]*, Theorem 8).*

Remark 8. *Our results contain those of* [20] *with the right choice of kernel* T*.*

Theorem 5. *Let* $f, g : [0, \infty) \to \mathbb{R}$*,* $f, g \in L_T^+[a, b]$ *such that* f *and* g *are differentiable with* f' *bounded below by* $m_1 = \inf_{t \in [0,\infty)} f'(t)$ *and* g' *bounded below by* $m_2 = \inf_{t \in [0,\infty)} g'(t)$*. Then we have*

$$\begin{aligned} & J_{T,a^+}(h_1 h_2)(b) \\ & \geq \tau(b-a)^{-1} J_{T,a^+}(f)(b)\, J_{T,a^+}(g)(b) - \frac{m_2}{\tau(b-a)}\, J_{T,a^+}(f)(b)\, J_{T,a^+}(t)(b) \\ & \quad - \frac{m_1}{\tau(b-a)}\, J_{T,a^+}(g)(b)\, J_{T,a^+}(t)(b) + \frac{m_1 m_2}{\tau(b-a)}\, J_{T,a^+}(t)(b)\, J_{T,a^+}(t)(b) \\ & \quad + m_2\, J_{T,a^+}(tf)(b) + m_1\, J_{T,a^+}(tg)(b) - m_1 m_2\, J_{T,a^+}\left(t^2\right)(b), \end{aligned}$$

where $t(x) = x$ *is the identity function.*

Proof. Let $p_1(x) = m_1 x$ and $h_1(x) = f(x) - p_1(x)$, similarly, $p_2(x) = m_2 x$ and $h_2(x) = g(x) - p_2(x)$. Since h_1 and h_2 is differentiable and increasing on $[0, \infty)$, applying (2) gives us

$$\begin{aligned} & J_{T,a^+}(h_1 h_2)(b) \\ & \geq [\tau(b-a)]^{-1} J_{T,a^+}(h_1)(b)\, J_{T,a^+}(h_2)(b) \\ & \geq [\tau(b-a)]^{-1} \left[J_{T,a^+}(f)(b) - J_{T,a^+}(p_1)(b)\right] \left[J_{T,a^+}(g)(b) - J_{T,a^+}(p_2)(b)\right] \\ & \geq \tau(b-a)^{-1} J_{T,a^+}(f)(b)\, J_{T,a^+}(g)(b) - \frac{m_2}{\tau(b-a)}\, J_{T,a^+}(f)(b)\, J_{T,a^+}(t)(b) \\ & \quad - \frac{m_1}{\tau(b-a)}\, J_{T,a^+}(g)(b)\, J_{T,a^+}(t)(b) + \frac{m_1 m_2}{\tau(b-a)}\, J_{T,a^+}(t)(b)\, J_{T,a^+}(t)(b). \end{aligned} \tag{9}$$

Moreover,

$$J_{T,a^+}(h_1 p_2)(b) = m_2\, J_{T,a^+}(t h_1)(b) = m_2\, J_{T,a^+}(tf)(b) - m_1 m_2\, J_{T,a^+}\left(t^2\right)(b) \tag{10}$$

similarly,

$$J_{T,a^+}(h_2 p_1)(b) = m_1\, J_{T,a^+}(tg)(b) - m_1 m_2\, J_{T,a^+}\left(t^2\right)(b) \tag{11}$$

and

$$J_{T,a^+}(p_1 p_2)(b) = m_1 m_2\, J_{T,a^+}\left(t^2\right)(b). \tag{12}$$

From the equality

$$fg = (h_1 + p_1)(h_2 + p_2) = h_1h_2 + h_1p_2 + h_2p_1 + p_1p_2$$

we have

$$J_{T,a^+}(fg)(b) = J_{T,a^+}(h_1h_2)(b) + J_{T,a^+}(h_1p_2)(b) + J_{T,a^+}(h_2p_1)(b) + J_{T,a^+}(p_1p_2)(b),$$

and this equality together with (9)–(12) implies the required result. □

Remark 9. *In case of $m_1 = 0$, we obtain Theorem 4.*

Remark 10. *The results obtained in this work can be extended if we consider instead of f and g, $-f$ and g or f and $-g$, in the notion of synchronous functions, in which case the direction of the inequalities changes .*

3. Conclusions

In this work, we have obtained the Chebyshev inequality from Theorem 1 within the framework of generalized integrals. In addition to the observations made, which prove the strength of our results, we would like to present a couple of variants of the classic Chebyshev inequality.

If we take kernel $T = t^{\alpha}$, $\alpha < 1$, then we get

$$J_{T,a^+}(fg)(b) \geq \frac{1-\alpha}{(b-a)^{1-\alpha}} J_{T,a^+}(f)(b)\, J_{T,a^+}(g)(b).$$

In case of taking kernel $T = e^{\alpha t}$, $\alpha \neq 0$, then we have the following variant of the Chebyshev inequality:

$$J_{T,a^+}(fg)(b) \geq \frac{\alpha}{1-e^{-\alpha(b-a)}} J_{T,a^+}(f)(b)\, J_{T,a^+}(g)(b).$$

Author Contributions: P.M.G. and J.E.N.V. worked together in the initial formulation of the mathematical results. P.K. helped with additional mathematical content and presentation. All the authors provided critical work resulted in the final form of the manuscript. All authors have read and agreed to the published version of the manuscript.

Conflicts of Interest: The authors declare that they have no competing interests.

References

1. Chebyshev, P.L. Sur les expressions approximatives des integrales definies par les autres prises entre les mêmes limites. *Proc. Math. Soc. Charkov.* **1882**, *2*, 93–98.
2. Özdemir, M.E.; Set, E.; Akdemir, A.O.; Sarikaya, M.Z. Some new Chebyshev type inequalities for functions whose derivatives belongs to L_p spaces. *Afrika Mat.* **2015**, *26*, 1609–1619. [CrossRef]
3. Set, E.; Choi, J.; Mumcu, İ. Chebyshev type inequalities involving generalized Katugampola fractional integral operators. *Tamkang J. Math.* **2019**, *50*, 381–390. [CrossRef]
4. Set, E.; Dahmani, Z.; Mumcu, İ., New extensions of Chebyshev type inequalities using generalized Katugampola integrals via Pólya–Szegö inequality. *IJOCTA* **2018**, *8*, 137–144. [CrossRef]
5. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; North-Holland Mathematical Studies; Elsevier (North-Holland) Science Publishers: Amsterdam, The Netherlands, 2006; Volume 204.
6. Podlubny, I. *Fractional Differential Equations. An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and Some of Their Applications*; Academic Press: San Diego, CA, USA, 1999.
7. Belarbi, S.; Dahmani, Z. On some new fractional integral inequalities. *J. Inequal. Pure Appl. Math.* **2009**, *10*, 1–12.

8. Chen, F. Extensions of the Hermite-Hadamard inequality for convex functions via fractional integrals. *J. Math. Inequal.* **2016**, *10*, 75–81. [CrossRef]
9. Dahmani, Z. New inequalities in fractional integrals. *Int. J. Nonlinear Sci.* **2010**, *9*, 493–497.
10. Gorenflo, R.; Mainardi, F. *Fractional Calculus: Integral and Differential Equations of Fractional Order*; Springer: Vienna, Austria, 1997; pp. 223–276.
11. İşcan, İ. Hermite-Hadamard-Fejér type inequalities for convex functions via fractional integrals. *Stud. Univ. Babes-Bolyai Math.* **2015**, *60*, 355–366.
12. Machado, J.T.; Kiryakova, V.; Mainardi, F. Recent history of fractional calculus. *Commun. Nonlinear Sci. Numer. Simul.* **2011**, *16*, 1140–1153. [CrossRef]
13. Sarikaya, M.Z.; Set, E.; Yaldiz, H.; Başak, N. Hermite–Hadamard's inequalities for fractional integrals and related fractional inequalities. *Math. Comput. Model.* **2013**, *57*, 2403–2407. [CrossRef]
14. Set, E. New inequalities of Ostrowski type for mappings whose derivatives are s-convex in the second sense via fractional integrals. *Comput. Math. Appl.* **2012**, *63*, 1147–1154. [CrossRef]
15. Set, E.; İşcan, İ.; Zehir, F. On some new inequalities of Hermite-Hadamard type involving harmonically convex functions via fractional integrals. *Konuralp J. Math.* **2015**, *3*, 42–55.
16. Set, E.; Mumcu, İ.; Demirbaş, S. Conformable fractional integral inequalities of Chebyshev type. *RACSAM* **2019**, *113*, 2253–2259. [CrossRef]
17. Khan, M.A.; Khan, T.U. Parameterized Hermite-Hadamard Type Inequalities For Fractional Integrals. *Turkish J. Ineqal.* **2017**, *1*, 26–37.
18. Sarıkaya, M.Z.; Ertuğral, F. On the Generalized Hermite-Hadamard Inequalities. Available online: https://www.researchgate.net/publication/321760443 (accessed on 19 December 2019).
19. Yaldız, H.; Akdemir, A.O. Katugampola Fractional Integrals within the Class of Convex Functions. *Turk. J. Sci.* **2018**, *3*, 40–50.
20. Nisar, K.S.; Rahman, G.; Mehrez, K. Chebyshev type inequalities via generalized fractional conformable integrals. *J. Inequal. Appl.* **2019**, *2019*, 245. [CrossRef]