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Numerical Solution of Nonlinear Fractional Diffusion Equation in Framework of the Yang–Abdel–Cattani Derivative Operator

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Abstract: In this manuscript, the time-fractional diffusion equation in the framework of the Yang–Abdel–Cattani derivative operator is taken into account. A detailed proof for the existence, as well as the uniqueness of the solution of the time-fractional diffusion equation, in the sense of YAC derivative operator, is explained, and, using the method of α -HATM, we find the analytical solution of the time-fractional diffusion equation. Three cases are considered to exhibit the convergence and fidelity of the aforementioned α -HATM. The analytical solutions obtained for the diffusion equation using the Yang–Abdel–Cattani derivative operator are compared with the analytical solutions obtained using the Riemann–Liouville (RL) derivative operator for the fractional order $\gamma = 0.99$ (nearly 1) and with the exact solution at different values of t to verify the efficiency of the YAC derivative operator.

Keywords: fractional derivative; existence and uniqueness; α -homotopy analysis transform method

MSC: Primary 92B05, 92C60; Secondary 26A33



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1. Introduction

A parabolic partial differential equation that delineates the movement of energy and matter in a medium is called a diffusion equation. The diffusion process of heat or mass can be described using the concept of ordinary and partial derivatives. In today's era, fractional calculus [1–6] is emerging as an efficient and powerful tool in the field of science and technology. It is the branch of mathematics pertaining to the derivatives and integrals of arbitrary order and it is fruitful in explaining the concepts of damping, wave propagation and diffusion, biology, genetic algorithms, control systems, economy and finance, signal processing, robotics, system identification, electromagnetism, heat transfer, and many more. The literature is brimming with developments made in the field of fractional calculus. The most widely accepted definition including the singular kernel was proposed by Riemann and Liouville–Caputo. The next classification of fractional derivatives are made on the basis of the non-singular kernel. Caputo–Fabrizio proposed the first definition in the field of fractional calculus with a non-singular kernel [7]. Further development and success in the definitions of fractional derivatives were introduced by Prabhakar, Sonine, Wiman, Miller–Ross, Gorenglo, Mainardi, Mittag–Leffler, Atangana–Baleanu, Yang–Abdel–Cattani, and a lot more, who propose that the non-singular kernels are special functions, such as the Mittag–Leffler function, Miller–Ross function, Wiman function, Kohlrausch–William–Watts function, Rabotnov function, Prabhakar function, etc., see [8–13]. The fractional derivatives

in which the non-singular kernels are the special functions are called general fractional-order derivatives. Further applications of fractional derivatives can be seen in [14–27]. In this work, we will find the analytical solution of the following nonlinear fractional diffusion equation in the framework of the Yang–Abdel–Cattani (YAC) derivative operator.

$${}_0^{YAC} \mathcal{D}_t^\gamma(w(\eta, t)) = \frac{\partial}{\partial \eta} \left(w^a \frac{\partial w(\eta, t)}{\partial \eta} \right) \quad w(\eta, 0) = w_0, \quad (1)$$

where ${}_0^{YAC} \mathcal{D}_t^\gamma(w(\eta, t))$ represents the Yang–Abdel–Cattani (YAC) fractional derivative of $w(\eta, t)$, w is the density of the diffusing medium at point η and at time t . This generalized fractional derivative was presented by Yang et al. with the Rabotnov exponential function as the non-singular kernel. In this work, we will present a detailed proof for the existence as well as the uniqueness of the solution of the time-fractional diffusion equation in the framework of the YAC derivative operator and, using the α -homotopy analysis transform method, we will find the analytical solution of the fractional diffusion equation in the sense of the YAC derivative operator. The analytical solutions obtained for the fractional diffusion equation using the YAC derivative operator are compared to the analytical solution obtained using the Riemann–Liouville (RL) derivative operator for the fractional order $\gamma = 0.99$ (nearby 1) and with the exact solution at different values of t to verify the efficiency of the YAC fractional derivative operator. Graphical representations of the analytical solutions are also given for a better understanding of the α -HATM.

2. Definitions

Definition 1. [13] We define $\gamma, \chi \in \mathbb{R}^+$; the following series defines the Rabotnov exponential function of order γ

$$\Psi(\chi u^\gamma) = \sum_{s=0}^{\infty} \frac{\chi^s u^{(s+1)(\gamma+1)-1}}{\Gamma((s+1)(\gamma+1))}, \quad u \in \mathbb{C} \quad (2)$$

Definition 2. [13] For k on $L^1[a, b]$, $t > 0$, $\chi \in \mathbb{R}^+$, $0 < \gamma \leq 1$, the following defines the Yang–Abdel–Cattani fractional derivative of order γ

$${}_0^{YAC} \mathcal{D}_t^\gamma(k(t)) = \int_0^t \Psi_\gamma(-\chi(t-\rho)^\gamma) k'(\rho) d\rho. \quad (3)$$

where, Ψ_γ represents the Rabotnov exponential function of order γ .

Definition 3. [13] The following defines the Laplace transform for the Yang–Abdel–Cattani fractional derivative

$$\mathcal{L}({}_0^{YAC} \mathcal{D}_t^\gamma(k(t))) = \frac{1}{p^{\gamma+1}} \frac{p\mathcal{L}[k(t)] - k(0)}{1 + \chi p^{-(\gamma+1)}} \quad (4)$$

Definition 4. [13] For $k \in L^1[a, b]$, $t > 0$, $0 < \gamma \leq 1$, $\chi \in \mathbb{R}^+$, the following defines the fractional integral with Rabotnov fractional exponential function, of order γ

$$I_{YAC}^\gamma k(t) = \int_0^t \Psi_\gamma(-\chi(t-\rho)^\gamma) k(\rho) d\rho. \quad (5)$$

Definition 5. [13] The following defines the Laplace transform for the Yang–Abdel–Cattani fractional integral

$$\mathcal{L}(I_{YAC}^\gamma k(t)) = \frac{1}{p^{\gamma+1}} \frac{\mathcal{L}(k(t))}{1 + \chi p^{-(\gamma+1)}} \quad (6)$$

3. Existence of Solution of Fractional Diffusion Equation Using Yang–Abdel–Cattani Derivative Operator

Theorem 1. Let us assume that the function $f(\eta, t, w, w', w'')$ satisfies the Lipschitz condition as

$$|f(\eta, t, w, w', w'') - f(\eta, t, w_1, w'_1, w''_1)| \leq M_1|w - w_1| + M_2|w' - w'_1| + M_3|w'' - w''_1|. \quad (7)$$

We also assume that

$$|w' - w'_1| \leq k_1|w - w_1|$$

$$|w'' - w''_1| \leq k_2|w - w_1|$$

where $k_1, k_2 \in \mathbb{R}^+$ then there exists a unique solution for the following time-fractional differential equation.

$${}_0^{YAC}D_t^\gamma(w(\eta, t)) = \frac{\partial}{\partial \eta} \left(w^a \frac{\partial w(\eta, t)}{\partial \eta} \right) \quad (8)$$

Proof. We define

$$\Phi(w, \eta) = f(\eta, t, w, w', w'') = \frac{\partial}{\partial \eta} \left(w^a \frac{\partial w(\eta, t)}{\partial \eta} \right) \quad (9)$$

We first show that $\Phi(w, \eta)$ satisfies Lipschitz condition. Consider

$$\begin{aligned} ||\Phi(w, \eta) - \Phi(w_1, \eta)|| &= ||f(\eta, t, w, w', w'') - f(\eta, t, w_1, w'_1, w''_1)|| \\ &\leq M_1||w - w_1|| + M_2||w' - w'_1|| + M_3||w'' - w''_1|| \\ &\leq M_1||w - w_1|| + M_2k_1||w - w_1|| + M_3k_2||w - w_1|| \\ &= (M_1 + M_2k_1 + M_3k_2)||w - w_1||. \end{aligned} \quad (10)$$

We also define $M_1 + M_2k_1 + M_3k_2 = M$. So finally, we have

$$||\Phi(w, \eta) - \Phi(w_1, \eta)|| = ||f(\eta, t, w, w', w'') - f(\eta, t, w_1, w'_1, w''_1)|| \quad (11)$$

$$\leq M||w - w_1|| \quad (12)$$

Using Picard's theorem, we obtain

$$w(\eta, t) = w(\eta, 0) + \int_0^t \Psi_\gamma(-\chi(t - \rho)^\gamma) \Phi(w, \eta(\rho)) d\rho. \quad (13)$$

For convenience, we write

$$\int_0^t \Psi_\gamma(-\chi(t - \rho)^\gamma) \Phi(w, \eta(\rho)) d\rho = I_{YAC}^\gamma \Phi(w, \eta)$$

Finally, we have

$$w(\eta, t) = w(\eta, 0) + I_{YAC}^\gamma \Phi(w, \eta) \quad (14)$$

$$w(\eta, t) - w(\eta, 0) = I_{YAC}^\gamma \Phi(w, \eta) \quad (15)$$

$$||w(\eta, t) - w(\eta, 0)|| = ||I_{YAC}^\gamma \Phi(w, \eta)|| \quad (16)$$

$$= \left\| \int_0^t \Psi_\gamma(-\chi(t - \rho)^\gamma) \Phi(w, \eta(\rho)) d\rho \right\| \quad (17)$$

$$\leq \int_0^t ||\Psi_\gamma(-\chi(t - \rho)^\gamma)|| ||\Phi(w, \eta(\rho))|| d\rho. \quad (18)$$

$$\leq ||\Phi(w, \eta(\rho))|| \int_0^t \Psi_\gamma(-\chi(t - \rho)^\gamma) \quad (19)$$

$$= I_{YAC}^\gamma(1) ||\Phi(w, \eta(\rho))|| \quad (20)$$

As we proved that $\Phi(w, \eta)$ satisfies Lipschitz condition, so the following holds

$$\|\Phi(w, \eta)\| \leq K \quad (21)$$

Hence

$$\|w(\eta, t) - w(\eta, 0)\| \leq KI_{YAC}^\gamma(1) \quad (22)$$

Finally, we consider

$$\|w(\eta, t) - w_1(\eta, t)\| = \|I_{YAC}^\gamma \Phi(w, \eta) - I_{YAC}^\gamma \Phi(w_1, \eta)\| \quad (23)$$

$$\leq I_{YAC}^\gamma(1) \|\Phi(w, \eta) - \Phi(w_1, \eta)\| \quad (24)$$

$$\leq I_{YAC}^\gamma M \|w - w_1\| \quad (25)$$

For the above map to be a contraction, we must have

$$M I_{YAC}^\gamma \leq 1 \quad (26)$$

$$I_{YAC}^\gamma \leq \frac{1}{M} \quad (27)$$

Hence the existence and the uniqueness of the solution follows as a consequence of the Banach fixed point theorem. \square

4. α -HATM Solution of Nonlinear Time-Fractional Diffusion Equation

Consider the nonlinear fractional diffusion equation, given as:

$${}_0^{YAC} \mathcal{D}_t^\gamma(w(\eta, t)) = \frac{\partial}{\partial \eta} \left(w^a \frac{\partial w(\eta, t)}{\partial \eta} \right) \quad (28)$$

We rewrite the above equation as

$${}_0^{YAC} \mathcal{D}_t^\gamma(w(\eta, t)) - \left[a w^{a-1} \left(\frac{\partial w}{\partial \eta} \right)^2 + w^a \frac{\partial^2 w}{\partial \eta^2} \right] = 0 \quad (29)$$

Taking the Laplace transform on the two sides of Equation (29),

$$\frac{1}{p^{\gamma+1}} \left[\frac{p \mathcal{L}[w(\eta)] - w(\eta, 0)}{1 + \chi p^{-(\gamma+1)}} \right] - \mathcal{L} \left[a w^{a-1} \left(\frac{\partial w}{\partial \eta} \right)^2 + w^a \frac{\partial^2 w}{\partial \eta^2} \right] = 0 \quad (30)$$

By simplifying, we obtain

$$\mathcal{L}[w(\eta, t)] - \frac{w(\eta, 0)}{p} - \left(\frac{p^{\gamma+1} + \chi}{p} \right) \mathcal{L} \left[a w^{a-1} \left(\frac{\partial w}{\partial \eta} \right)^2 + w^a \frac{\partial^2 w}{\partial \eta^2} \right] = 0 \quad (31)$$

Let \mathcal{N} be a nonlinear operator, defined as

$$\mathcal{N}[\beta(\eta, t, \alpha)] = \mathcal{L}[\beta(\eta, t, \alpha)] - \frac{w(\eta, 0)}{p} - \left(\frac{p^{\gamma+1} + \chi}{p} \right) \mathcal{L} \left[a \beta^{a-1} \left(\frac{\partial \beta}{\partial \eta} \right)^2 + \beta^a \frac{\partial^2 \beta}{\partial \eta^2} \right] \quad (32)$$

where $\beta(\eta, t, \alpha)$ is a function in η, t, α , and $\alpha \in [0, 1/r]$ is an embedding parameter. Now construct the homotopy as

$$(1 - r\alpha)L[\beta(\eta, t, \alpha) - w_0(\eta, t)] = \alpha k \mathcal{N}[\beta(\eta, t, \alpha)] \quad (33)$$

where $\beta(\eta, t, \alpha)$ is a function of η, t and α , $w_0(\eta, t)$ is an initial guess of $w(\eta, t)$, \mathcal{L} is the Laplace transform, and $k \neq 0$ is an auxiliary parameter. From above equation, we see that

when $\alpha = 0$, $\beta(\eta, t, 0) = w_0(\eta, t)$

when $\alpha = \frac{1}{r}$, $\beta(\eta, t, 1/r) = w(\eta, t)$

This shows that as α varies from 0 to $1/r$, the solution $\beta(\eta, t, \alpha)$ changes from $w_0(\eta, t)$ the initial guess to $w(\eta, t)$ the exact solution.

Expand $\beta(\eta, t, \alpha)$ with respect to α using the Taylor series, we obtain

$$\beta(\eta, t, \alpha) = w_0(\eta, t) + \sum_{i=1}^{\infty} w_i(\eta, t) \alpha^i \quad (34)$$

where

$$w_i(\eta, t) = \frac{1}{i!} \left. \frac{\partial^i \beta(\eta, t; \alpha)}{\partial \alpha^i} \right|_{\alpha=0} \quad (35)$$

Let $w_0(\eta, t)$, k , r be selected appropriately, the series defined in Equation (35) converges at $\alpha = 1/r$, hence

$$w(\eta, t) = w_0(\eta, t) + \sum_{i=1}^{\infty} w_i(\eta, t) \left(\frac{1}{r} \right)^i \quad (36)$$

Defining the vectors $\bar{w}_n = \{w_0, w_1, \dots, w_n\}$, and differentiating Equation (33) i -times with respect to α , and substituting $\alpha = 0$, and lastly dividing them by $i!$, we obtain:

$$\mathcal{L}[w_i(\eta, t) - \zeta_i w_{i-1}(\eta, t)] = k \mathfrak{R}_i(w_{i-1}(\eta, t)), \quad (37)$$

where

$$\mathfrak{R}_i(w_{i-1}(\eta, t)) = \frac{1}{(i-1)!} \left. \frac{\partial^{i-1} N(\eta, t; \alpha)}{\partial \alpha^{i-1}} \right|_{\alpha=0} \quad (38)$$

and

$$\zeta_i = \begin{cases} 0, & i \leq 1 \\ r, & \text{otherwise.} \end{cases} \quad (39)$$

Using Equations (32) and (38), we obtain

$$\begin{aligned} \mathfrak{R}_i(w_{i-1}(\eta, t)) = L[w_{i-1}] - \frac{w(\eta, 0)}{p} \left(1 - \frac{\zeta_i}{r} \right) + \\ \mathcal{L} \left[a w_{i-1}^{a-1} \left(\frac{\partial w_{i-1}}{\partial \eta} \right)^2 + w_{i-1}^a \frac{\partial^2 w_{i-1}}{\partial \eta^2} \right] \end{aligned} \quad (40)$$

Lastly, take the inverse Laplace transform on both sides of Equation (37),

$$w_m(\eta, t) = \zeta_i w_{i-1}(\eta, t) + k \mathcal{L}^{-1}[\mathfrak{R}_i(w_{i-1})]. \quad (41)$$

Finally, opting for the suitable values of k and r , the α -HATM series solution is obtained, which is given as

$$w(\eta, t) = \lim_{N \rightarrow \infty} \sum_{i=0}^{\infty} w_i(\eta, t) \left(\frac{1}{r} \right)^i \quad (42)$$

We now consider different cases of the above nonlinear diffusion equation.

4.1. Case 1

We will now find the analytical solution of the following non-linear fractional diffusion equation in sense of YAC derivative operator using above mentioned α -HATM.

$${}_0^{YAC} \mathcal{D}_t^\gamma (v(\eta, t)) = \frac{\partial}{\partial \eta} \left(\frac{\partial v(\eta, t)}{\partial \eta} \right), \quad 0 < t < 1; \quad 0 < \gamma < 1, \quad v(\eta, 0) = \sin(\pi \eta) \quad (43)$$

Using the α -HATM, the series solution is given as

$$v_0^{YAC}(\eta, t) = \sin(\pi\eta) \quad (44)$$

$$v_1^{YAC}(\eta, t) = k\pi^2 \left(t\chi + \frac{t^{-\gamma}}{\Gamma(1-\gamma)} \right) \sin(\pi\eta) \quad (45)$$

$$v_2^{YAC}(\eta, t) = rv_1^{YAC}(\eta, t) + \frac{1}{2}k^2\pi^2 \left[\frac{2\pi^2 t^{-2\gamma}}{\Gamma(1-2\gamma)} + \frac{2t^{-\gamma}}{\Gamma(1-\gamma)} t\chi \left(2 + \pi^2 t\chi + \frac{4\pi^2 t^{-\chi}}{\Gamma(2-\gamma)} \right) \right] \sin(\pi\eta) \quad (46)$$

The following diffusion equation's

$$\frac{\partial v(\eta, t)}{\partial t} = \frac{\partial^2 v(\eta, t)}{\partial \eta^2} \quad (47)$$

exact solution is given as

$$\sin(\pi\eta)e^{-\pi^2 t} \quad (48)$$

In Table 1, we will compare the α -HATM solutions obtained for Case 1 in terms of the YAC fractional derivative operator with the analytical solutions obtained using the Riemann–Liouville fractional derivative operator [28] and with the considered diffusion equation's exact solution at different values of t for $\eta = 0.25$ and $\gamma = 0.99$ (γ close to 1) $k = -0.01$, $r = 3$ and $\chi = 3$.

Table 1. Comparison of α -HATM Solution in sense of YAC derivative operator, RL derivative operator and exact solution for Case 1.

t	$v(RL)$	$v(YAC)$	$v(Exact)$	$Error(RL)$	$Error(YAC)$
0.1	0.196571	0.263464	0.263544	6.6973×10^{-2}	8×10^{-5}
0.2	0.902091	0.0981688	0.098225	8.0159×10^{-3}	5.62×10^{-5}
0.3	0.0303086	0.0361351	0.0366092	6.3006×10^{-3}	4.741×10^{-4}
0.4	0.0165324	0.0131774	0.0136445	2.8879×10^{-3}	4.671×10^{-4}
0.5	0.00463999	0.00507924	0.00508543	4.4544×10^{-4}	6.19×10^{-6}
0.6	0.00134119	0.00188248	0.00189538	5.5419×10^{-4}	1.29×10^{-5}
0.7	0.000591754	0.000645395	0.000706423	1.14669×10^{-4}	6.1028×10^{-5}
0.8	0.000208632	0.000240379	0.000263289	5.4657×10^{-5}	2.291×10^{-5}
0.9	0.0000935588	0.0000971667	0.00009813	4.5712×10^{-6}	9.633×10^{-7}
1.0	0.0000300947	0.0000351593	0.0000365738	6.4791×10^{-6}	1.4145×10^{-6}

In Figure 1, the 3-D plots of the α -HATM solution for the YAC operator are compared to the exact solution for $\gamma = 0.99$ (γ nearby 1).

4.2. Case 2

We will now find the analytical solution of the following non-linear fractional diffusion equation in sense of YAC derivative operator using above mentioned α -HATM.

$${}_0^{YAC} \mathcal{D}_t^\gamma (v(\eta, t)) = \frac{\partial}{\partial \eta} \left(\frac{v^2 \partial v(\eta, t)}{\partial \eta} \right), \quad 0 < \gamma < 1; \quad 0 < t < 1, \quad v(\eta, 0) = \frac{\eta + b}{c} \quad (49)$$

Using the α -HATM, the series solution is given as

$$v_0^{YAC}(\eta, t) = \frac{\eta + b}{c} \quad (50)$$

$$v_1^{YAC}(\eta, t) = \frac{-k(b + \eta)}{4c^3} \left(t\chi + \frac{t^{-\gamma}}{\Gamma(1 - \gamma)} \right) \quad (51)$$

$$v_2^{YAC}(\eta, t) = rv_1^{YAC}(\eta, t) + \frac{k^2 t^{-2\gamma}(b + \eta)}{4c^5 \Gamma(1 - 2\gamma) \Gamma(1 - \gamma) \Gamma(2 - \gamma)} \left[2\Gamma(1 - \gamma) \Gamma(2 - \gamma) + t^\gamma \Gamma(1 - 2\gamma) (-c^2 \Gamma(1 - 2\gamma) + t\chi \Gamma(1 - \gamma) [4 - t^\gamma (c^2 - t\chi) \Gamma(2 - \gamma)]) \right] \quad (52)$$

The following diffusion equation's

$$\frac{\partial v(\eta, t)}{\partial t} = \frac{\partial}{\partial \eta} \left(\frac{v^2 \partial v(\eta, t)}{\partial \eta} \right) \quad (53)$$

exact solution is given as

$$\frac{\eta + b}{2\sqrt{c^2 - t}}, \quad t \leq c^2 \quad (54)$$

In Table 2, we will compare the α -HATM solutions obtained for Case 2 in terms of the YAC fractional derivative operator with the analytical solutions obtained in terms of the Riemann–Liouville fractional derivative operator [28] and with the considered diffusion equation's exact solution at different values of t for $k = -0.19$, $r = 1$, $\chi = 1$, $b = 2$, $c = 3$, $\eta = 0.25$ and $\gamma = 0.99$.

Table 2. Comparison of α -HATM Solution in sense of YAC derivative operator, RL derivative operator and exact solution for Case 2.

t	v (RL)	v (YAC)	v (Exact)	Error (RL)	Error (YAC)
0.1	0.362916	0.377039	0.377101	1.4185×10^{-2}	6.2×10^{-5}
0.2	0.375416	0.377759	0.379237	3.821×10^{-3}	1.478×10^{-3}
0.3	0.379474	0.381023	0.381411	1.937×10^{-3}	3.88×10^{-4}
0.4	0.380159	0.383075	0.383622	3.463×10^{-3}	5.47×10^{-4}
0.5	0.382443	0.384292	0.3858872	3.4442×10^{-3}	1.58×10^{-3}
0.6	0.387566	0.388041	0.388162	5.96×10^{-4}	1.21×10^{-4}
0.7	0.388769	0.390407	0.390493	1.724×10^{-3}	8.6×10^{-5}
0.8	0.39034	0.392672	0.392867	2.527×10^{-3}	1.72×10^{-4}
0.9	0.390757	0.395034	0.395285	4.528×10^{-3}	2.51×10^{-4}
1.0	0.405465	0.397722	0.397748	7.717×10^{-3}	2.6×10^{-5}

In Figure 2, the 3-D plots of the α -HATM solution for the YAC operator are compared to the exact solution for $\gamma = 0.99$ (γ nearby 1).

4.3. Case 3

We will now find the analytical solution of the following non-linear fractional diffusion equation in sense of YAC derivative operator using above mentioned α -HATM.

$${}_0^{YAC} \mathcal{D}_t^\gamma (v(\eta, t)) = \frac{\partial}{\partial \eta} \left(\frac{v^{-2} \partial v(\eta, t)}{\partial \eta} \right), \quad 0 < t < 1; \quad 0 < \gamma < 1, \quad v(\eta, 0) = \frac{1}{\sqrt{1 + \eta^2}} \quad (55)$$

Using the α -HATM, the series solution is given as

$$v_0^{YAC}(\eta, t) = \frac{1}{\sqrt{1 + \eta^2}} \quad (56)$$

$$v_1^{YAC}(\eta, t) = -k \left[\frac{\eta^2}{(1 + \eta^2)^{3/2}} - \frac{1}{\sqrt{1 + \eta^2}} \right] \quad (57)$$

The following diffusion equation's

$$\frac{\partial v(\eta, t)}{\partial t} = \frac{\partial}{\partial \eta} \left(\frac{v^{-2} \partial v(\eta, t)}{\partial \eta} \right) \quad (58)$$

exact solution is

$$\frac{1}{\sqrt{\eta^2 + e^{2t}}} \quad (59)$$

In Table 3, we will compare the α -HATM solutions obtained for Case 3 in terms of the YAC fractional derivative operator with the analytical solutions obtained in terms of the Riemann–Liouville fractional derivative operator [28] and with the considered diffusion equation's exact solution at different values of t for $k = -0.6$, $r = 1$, $\chi = 1$, $\eta = 0.25$, and $\gamma = 0.99$.

Table 3. Comparison of α -HATM Solution in sense of YAC derivative operator, RL derivative operator and exact solution for Case 3.

t	$v(RL)$	$v(YAC)$	$v(Exact)$	$Error(RL)$	$Error(YAC)$
0.1	0.842589	0.861515	0.882539	3.995×10^{-2}	2.1024×10^{-2}
0.2	0.780412	0.805378	0.802101	2.1689×10^{-2}	3.277×10^{-3}
0.3	0.698454	0.723338	0.728431	2.9972×10^{-2}	5.093×10^{-3}
0.4	0.640023	0.64123	0.661101	2.1078×10^{-2}	1.9871×10^{-2}
0.5	0.589874	0.559094	0.599676	9.802×10^{-3}	4.058×10^{-3}
0.6	0.521176	0.553665	0.543718	2.254×10^{-2}	9.947×10^{-3}
0.7	0.462397	0.484287	0.492802	3.0405×10^{-2}	8.515×10^{-3}
0.8	0.460085	0.45875	0.446521	1.3564×10^{-2}	1.23×10^{-2}
0.9	0.428753	0.394834	0.404486	2.4267×10^{-2}	9.652×10^{-3}
1.0	0.387562	0.330926	0.366333	2.1229×10^{-2}	3.24×10^{-2}

In Figure 3, the 3-D plots of the α -HATM solution for the YAC operator are compared to the exact solution for $\gamma = 0.99$ (γ nearby 1).

5. Conclusions

In this paper, the time-fractional nonlinear diffusion equation is taken into consideration in regards to the Yang–Abdel–Cattani fractional derivative operator. The α -HATM is used to find the analytical solution of the nonlinear fractional diffusion equation. The analytical solutions obtained from the α -HATM, in the sense of the YAC derivative operator, are compared to the analytical solutions obtained via the Riemann–Liouville derivative operator and with the exact solution for all the three cases at distinct values of time t , and we observe that the analytical solutions obtained using the YAC derivative operator coincide with the exact solution more closely as compared to the Riemann–Liouville derivative operator when the value of fractional order is close to 1, i.e., $\gamma = 0.99$. In Figures 1–3 we plotted the 3-D representations of the solution $v(\eta, t)$ with the exact solution for case 1,

case 2, and case 3, respectively. In Figures 1–3a, the 3-D representation of the analytical solution attained using α -HATM along with the exact solution is given. In Figures 1–3b, the α -HATM solutions are plotted for different values of k along with the exact solution and we see that the α -HATM solution is in best fit with the exact solution for a suitable value of k and hence $k = -0.01$ works as an optimal value for case 1, $k = -0.19$ works as an optimal value for case 2, and $k = -0.6$ works as an optimal value for case 3. In Figures 1–3c, the α -HATM solutions are plotted for different values of r along with the exact solution and we see that the α -HATM solution is in best fit with the exact solution for a suitable value of r and hence $r = 3$ works as an optimal value for case 1 and $r = 1$ works as an optimal value for cases 2 and 3. In Figures 1–3d, the α -HATM solutions are plotted for different values of χ along with the exact solution and we see that the α -HATM solution is in best fit with the exact solution for a suitable value of χ and hence $\chi = 3$ works as an optimal value for case 1 and $\chi = 1$ works as optimal value for case 2 and 3. Hence, we see the efficient role of the parameters k , r provided in the α -HATM, and χ , given in the YAC definition in regulating the convergence of the solution. Hence we see that YAC fractional derivative operator provides better results than the Riemann–Liouville fractional derivative operator.

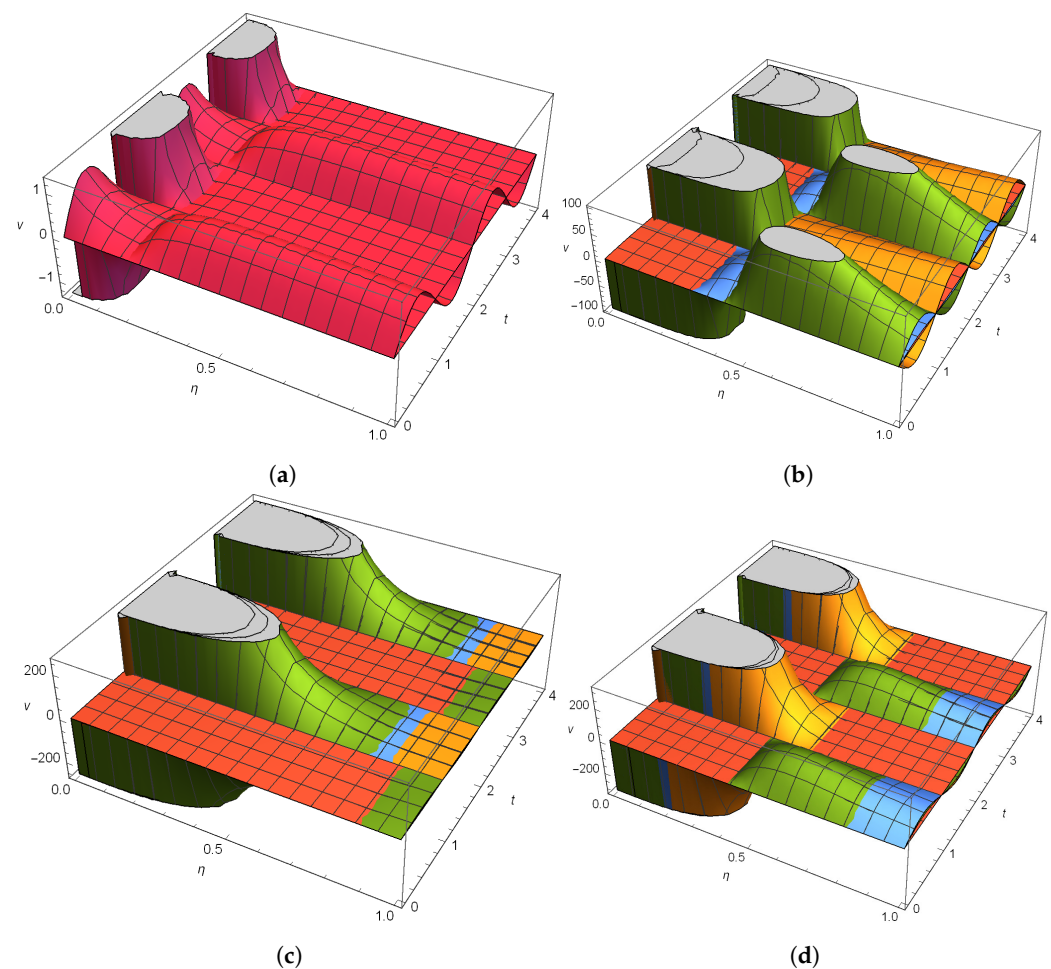


Figure 1. 3-D Plot of α -HATM solution for YAC operator with exact solution for case 1 when $\gamma = 0.99$. (a) α -HATM solution with exact solution. (b) α -HATM solution with exact solution for distinct values of k , Yellow ($k = -0.01$), Blue ($k = -0.04$), Green ($k = -0.08$), Red (Exact Solution). (c) α -HATM solution with exact solution for distinct values of r , Yellow ($r = 1$), Blue ($r = 2$), Green ($r = 3$), Red (Exact Solution). (d) α -HATM solution with exact solution for distinct values of χ , Yellow ($\chi = 3$), Blue ($\chi = 5$), Green ($\chi = 7$), Red (Exact Solution).

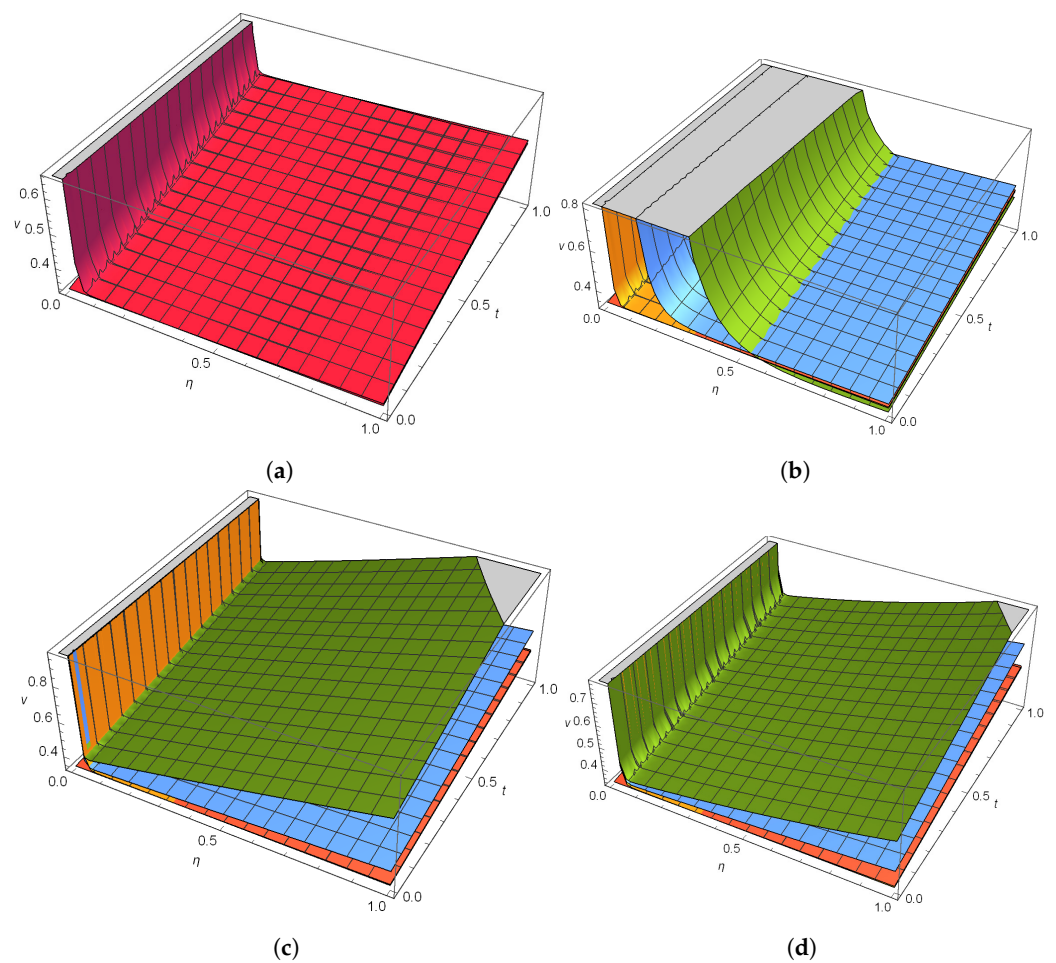


Figure 2. 3-D Plot of α -HATM solution for YAC operator with exact solution for case 2 when $\gamma = 0.99$. (a) α -HATM solution with exact solution. (b) α -HATM solution with the exact solution for distinct values of k , Yellow ($k = -0.19$), Blue ($k = -1$), Green ($k = -2$), Red (Exact Solution). (c) α -HATM solution with exact solution for distinct values of r , Yellow ($r = 1$), Blue ($r = 2$), Green ($r = 3$), Red (Exact Solution). (d) α -HATM solution with exact solution for distinct values of χ , Yellow ($\chi = 1$), Blue ($\chi = 5$), Green ($\chi = 10$), Red (Exact Solution).

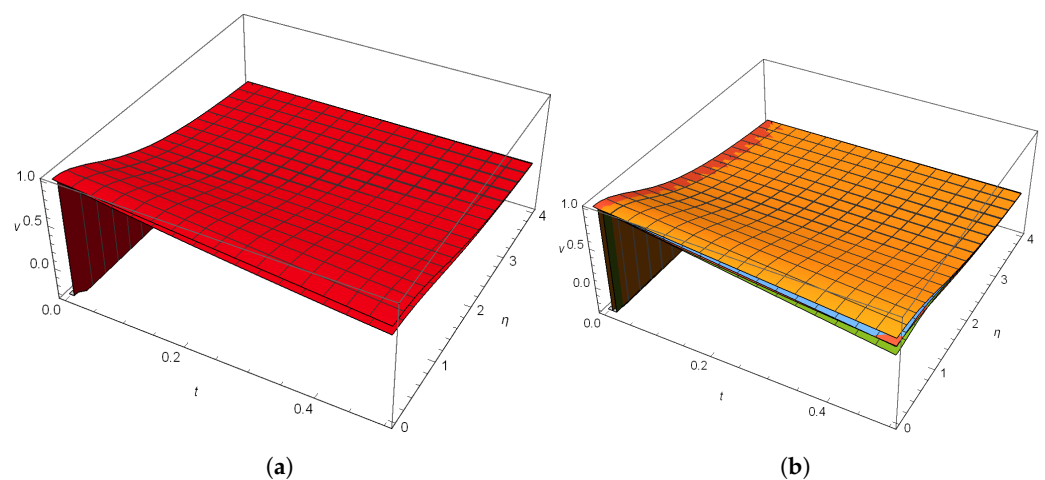


Figure 3. Cont.

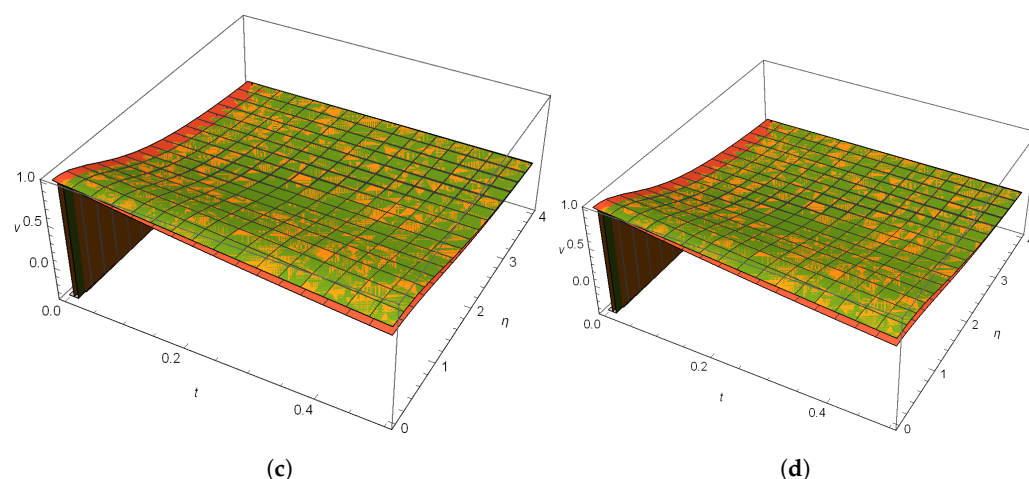


Figure 3. 3-D Plot of α -HATM solution for YAC operator with exact solution for case 3 when $\gamma = 0.99$. (a) α -HATM solution with exact solution. (b) α -HATM solution with exact solution for distinct values of k , Yellow ($k = -0.6$), Blue ($k = -0.8$), Green ($k = -1$), Red (Exact Solution). (c) α -HATM solution with exact solution for distinct values of r , Yellow ($r = 1$), Blue ($r = 2$), Green ($r = 3$), Red (Exact Solution). (d) α -HATM solution with exact solution for distinct values of χ , Yellow ($\chi = 1$), Blue ($\chi = 2$), Green ($\chi = 3$), Red (Exact Solution).

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