fractal and fractional

Article

Corrected Dual-Simpson-Type Inequalities for Differentiable Generalized Convex Functions on Fractal Set

Abdelghani Lakhdari [1], Wedad Saleh [2,*], Badreddine Meftah [3] and Akhlad Iqbal [4]

[1] Department CPST, Higher School of Industrial Technologies, P.O. Box 218, Annaba 23000, Algeria
[2] Department of Mathematics, Taibah University, Al-Medina 42353, Saudi Arabia
[3] Department of Mathematics, 8 May 1945 University, Guelma 24000, Algeria
[4] Department of Mathematics, Aligarh Muslim University, Aligarh 202002, India
* Correspondence: wlehabi@taibahu.edu.sa

Abstract: The present paper provides several corrected dual-Simpson-type inequalities for functions whose local fractional derivatives are generalized convex. To that end, we derive a new local fractional integral identity as an auxiliary result. Using this new identity along with generalized Hölder's inequality and generalized power mean inequality, we establish some new variants of fractal corrected dual-Simpson-type integral inequalities. Furthermore, some applications for error estimates of quadrature formulas as well as some special means involving arithmetic and p-logarithmic mean are offered to demonstrate the efficacy of our findings.

Keywords: corrected dual-Simpson inequality; local fractional derivatives; local fractional integrals; generalized convex functions; fractal set

Citation: Lakhdari, A.; Saleh, W.; Meftah, B.; Iqbal, A. Corrected Dual-Simpson-Type Inequalities for Differentiable Generalized Convex Functions on Fractal Set. *Fractal Fract.* **2022**, *6*, 710. https://doi.org/10.3390/fractalfract6120710

Academic Editor: Maria Rosaria Lancia

Received: 6 November 2022
Accepted: 24 November 2022
Published: 29 November 2022

1. Introduction

Numerous branches of mathematics and related disciplines, including economics, finance, and biology, heavily rely on the notion of convexity, which represents a potent technique for investigating a diverse range of unconnected topics in pure and applied sciences. This concept is strongly related to the development of the theory of inequalities which serves as an essential tool for studying certain properties of differential equation solutions and the error estimates of quadrature formulas.

In recent years, fractal analysis has led to a lot of new research. Gao-Yang-Kang's innovative and interesting idea of local fractional differentiation and integration has received a lot of attention from researchers. This idea has grown quickly because it can be used in many different ways, not just in mathematics but also in other fields of science. Ref. [1] looked into the local fractional-wave equation that is defined on Cantor sets. In [2], the heat-conduction equation in Cantor sets was given. Ref. [3] gives the perturbation solution for the oscillator with free-damped vibrations. In [4], the elliptic, hyperbolic, and parabolic fractional PDEs were looked at.

Regarding the integral inequalities in the fractal set via different kinds of generalized convexity, we mention: Hölder inequality [5], Hilbert inequality [6], Grüss inequality [7], Pompeiu inequality [8], Simpson's first formula [9,10], Simpson's second formula [11], Hermite–Hadamard type inequalities [9,12], Ostrowski's inequality [13–15], trapezium type inequality [12], generalized trapezium inequality [16], Féjer–Simpson inequality [17], Féjer inequalities [18], and Maclaurin type inequalities [19]. For more inequalities for generalized s-convex functions on fractal sets, see [20].

In this paper, we are concerned with three-point Newton-cotes formulas, of which the works listed below are examples.

The most renowned Newton–Cotes inequality involving three points is that of Simpson, which can be stated as follows:

$$\left|\frac{1}{6}\left(h(a)+4h\left(\frac{a+b}{2}\right)+h(b)\right)-\frac{1}{b-a}\int_a^b h(x)dx\right|\leq\frac{(b-a)^4}{2880}\left\|h^{(4)}\right\|_\infty,$$

where h is a four-times continuously differentiable function on (a,b), and

$$\left\|h^{(4)}\right\|_\infty=\sup_{x\in(a,b)}\left|h^{(4)}(x)\right|.$$

The aforementioned inequality is widely applied in the error estimation of Simpson's quadrature rule. More importantly, it is highly valued and studied by researchers.

In [21], Set et al. provided the following Simpson-type inequality for generalized quasi convex functions

Theorem 1. *Let $I\subseteq\mathbb{R}$ be an interval, $h:I^\circ\subseteq\mathbb{R}\to\mathbb{R}^\alpha$ (I° is the interior of I) such that $h\in D_\alpha(I^\circ)$ and $h^{(\alpha)}\in C_\alpha[a,b]$ for $a,b\in I^\circ$ with $a<b$. If $\left|h^{(\alpha)}\right|$ is a generalized quasi-convex function, then we have the inequality*

$$\begin{aligned}&\left|\left(\frac{1}{6}\right)^\alpha\left(h(a)+4^\alpha h\left(\frac{a+b}{2}\right)+h(b)\right)-\frac{\Gamma(\alpha+1)}{(b-a)^\alpha}\cdot{}_aI_b^{(\alpha)}h(x)\right|\\ \leq&(b-a)^\alpha\left(\frac{5}{18}\right)^\alpha\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\sup\left\{\left|h^{(\alpha)}(a)\right|,\left|h^{(\alpha)}(b)\right|\right\}.\end{aligned}$$

Moreover, Sarikaya et al. [10] presented the following Simpson-type inequality for generalized convex functions

Theorem 2. *Let $I\subseteq\mathbb{R}$ be an interval, $h:I^\circ\subseteq\mathbb{R}\to\mathbb{R}^\alpha$ (I° is the interior of I) such that $h\in D_\alpha(I^\circ)$ and $h^{(\alpha)}\in C_\alpha[a,b]$ for $a,b\in I^\circ$ with $a<b$. If $\left|h^{(\alpha)}\right|$ is a generalized convex function, then we have*

$$\begin{aligned}&\left|\left(\frac{1}{6}\right)^\alpha\left(h(a)+4^\alpha h\left(\frac{a+b}{2}\right)+h(b)\right)-\frac{\Gamma(\alpha+1)}{(b-a)^\alpha}\cdot{}_aI_b^{(\alpha)}h(x)\right|\\ \leq&\frac{(b-a)^\alpha}{12^\alpha}\left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}+\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right)\left(\left|h^{(\alpha)}(a)\right|+\left|h^{(\alpha)}(b)\right|\right).\end{aligned}$$

More importantly, Abdeljawad et al. [9] generalized the result obtained in [10] involving the generalized (s,m)-convexity

Theorem 3. *For $s,m\in(0,1]$, let $h:I^\circ\to\mathbb{R}^\alpha$ be a differentiable function on I° such that $h^{(\alpha)}\in C_\alpha[a,mb]$ for $a,b\in I^\circ$ with $a<b$. If $\left|h^{(\alpha)}\right|$ is generalized (s,m)-convex on I, then we have*

$$\begin{aligned}&\left|\left(\frac{1}{6}\right)^\alpha\left(h(a)+4^\alpha h\left(\frac{a+mb}{2}\right)+h(mb)\right)-\frac{\Gamma(\alpha+1)}{(mb-a)^\alpha}\cdot{}_aI_{mb}^{(\alpha)}h(x)\right|\\ \leq&(mb-a)^\alpha\left(\frac{2^\alpha\left(5^{(s+2)\alpha}-3^{(s+1)\alpha}\right)-5^\alpha\left(6^{(s+1)\alpha}+3^{(s+1)\alpha}\right)}{6^{(s+2)\alpha}}\right)\\ &\times\left(\frac{\Gamma(1+s\alpha)}{\Gamma(1+(s+1)\alpha)}+\frac{\Gamma(1+(s+1)\alpha)}{\Gamma(1+(s+2)\alpha)}\right)\left(\left|h^{(\alpha)}(a)\right|+m\left|h^{(\alpha)}(b)\right|\right).\end{aligned}$$

Furthermore, Du et al. [17] presented the following Simpson-like type inequality via generalized m-convexity

Theorem 4. *Let $h : I^\circ \to \mathbb{R}^\alpha$ be local fractional continuous such that $h^{(\alpha)} \in C_\alpha[a,b]$ with $0 \le a < mb$. If the mapping $|h^{(\alpha)}|^q$ for $q \ge 1$ is generalized m-convex on $[0,b]$ along with certain fixed $m \in (0,1]$, then the local fractional integral inequality stated below holds.*

$$\begin{aligned}
&\left|\left(\frac{1}{8}\right)^\alpha\left(h(a)+6h\left(\frac{a+mb}{2}\right)+h(mb)\right)-\frac{\Gamma(\alpha+1)}{(mb-a)^\alpha}\cdot {}_aI_{mb}^{(\alpha)}h(x)\right| \\
&\le\left(\frac{mb-a}{4}\right)^\alpha\left(\left(\frac{5}{8}\right)^\alpha\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\right)^{1-\frac{1}{q}} \\
&\times\left(\left(\left(\left(\frac{5}{64}\right)^\alpha\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}+\left(\frac{23}{64}\right)^\alpha\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\right)\left|h^{(\alpha)}(a)\right|^q\right.\right. \\
&\left.+m^\alpha\left(\left(-\frac{5}{64}\right)^\alpha\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}+\left(\frac{17}{64}\right)^\alpha\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\right)\left|h^{(\alpha)}(b)\right|^q\right)^{\frac{1}{q}} \\
&+\left(\left(\left(\frac{31}{64}\right)^\alpha\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}+\left(\frac{-7}{64}\right)^\alpha\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\right)\left|h^{(\alpha)}(a)\right|^q\right. \\
&\left.\left.+m^\alpha\left(\left(-\frac{31}{64}\right)^\alpha\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}+\left(\frac{47}{64}\right)^\alpha\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\right)\left|h^{(\alpha)}(b)\right|^q\right)^{\frac{1}{q}}\right).
\end{aligned}$$

Otherwise, in [22], the authors gave the so-called corrected dual-Simpson formula as follows

$$\int_a^b h(x)dx = \mathcal{S}(h) + R(h),$$

where

$$\mathcal{S}(h) = \frac{(b-a)}{15}\left(8h\left(\frac{3a+b}{4}\right) - h\left(\frac{a+b}{2}\right) + 8h\left(\frac{a+3b}{4}\right)\right)$$

and $R(h)$ denotes the associated approximation error.

Motivated by the above cited papers, in this work, we will discuss the corrected dual Simpson's formula given in [22] via local fractional integrals. To do so, we first establish a new integral identity. On the basis of this equality, we derive some corrected dual-Simpson-type inequalities for functions whose local fractional derivatives are generalized convex. In conclusion, some applications are provided.

2. Preliminaries

In this section, we recall some fractal theory concepts. For $0 < \alpha \le 1$, we have the following α-type sets:

The α-type set of integer is defined as:

$$\mathbb{Z}^\alpha := \{0^\alpha, \pm 1^\alpha, \pm 2^\alpha, \ldots, \pm n^\alpha, \ldots\}.$$

The definition of the α-type set of rational numbers is:

$$\mathbb{Q}^\alpha := \left\{a^\alpha = \left(\frac{b}{c}\right)^\alpha : b, c \in \mathbb{Z} \text{ and } c \neq 0\right\}.$$

The α-type irrational number set is defined as:

$$\mathbb{J}^\alpha := \left\{a^\alpha \neq \left(\frac{b}{c}\right)^\alpha : b, c \in \mathbb{Z} \text{ and } c \neq 0\right\}.$$

The α-type set of the real line numbers is defined as:

$$\mathbb{R}^{\alpha} := \mathbb{Q}^{\alpha} \cup \mathbb{J}^{\alpha}.$$

If the real line number set $\mathbb{R}^{\alpha}$ includes u^{α}, v^{α}, and w^{α}, then we have

- $u^{\alpha} + v^{\alpha}$ and $u^{\alpha}v^{\alpha}$ belongs the set $\mathbb{R}^{\alpha}$.
- $u^{\alpha} + v^{\alpha} = v^{\alpha} + u^{\alpha} = (u+v)^{\alpha} = (v+u)^{\alpha}$.
- $u^{\alpha} + (v^{\alpha} + w^{\alpha}) = (u+v)^{\alpha} + w^{\alpha}$.
- $u^{\alpha}v^{\alpha} = v^{\alpha}u^{\alpha} = (uv)^{\alpha} = (vu)^{\alpha}$.
- $u^{\alpha}(v^{\alpha}w^{\alpha}) = (u^{\alpha}v^{\alpha})w^{\alpha}$.
- $u^{\alpha}(v^{\alpha} + w^{\alpha}) = u^{\alpha}v^{\alpha} + u^{\alpha}w^{\alpha}$.
- $u^{\alpha} + 0^{\alpha} = 0^{\alpha} + u^{\alpha} = u^{\alpha}$ and $u^{\alpha}1^{\alpha} = 1^{\alpha}u^{\alpha} = u^{\alpha}$.

Gao-Yang-Kang [23,24] introduced the idea of the local fractional derivative and local fractional integral.

Definition 1 ([23])**.** *A non-differentiable function $h : \mathbb{R} \to \mathbb{R}^{\alpha}$ is local fractional continuous at x_0, if*

$$\forall \epsilon > 0, \exists \delta > 0 : |h(x) - h(x_0)| < \epsilon^{\alpha}$$

holds for $|x - x_0| < \delta$, where $\epsilon, \delta \in \mathbb{R}$.
$C_{\alpha}(a,b)$ denotes the set of all locally fractional continuous functions on (a,b).

Definition 2 ([23])**.** *At $x = x_0$, the local fractional derivative of $h(x)$ of order α is defined as follows:*

$$h^{(\alpha)}(x_0) = \left.\frac{d^{\alpha}h(x)}{dx^{\alpha}}\right|_{x=x_0} = \lim_{x \to x_0} \frac{\Delta^{\alpha}(h(x) - h(x_0))}{(x - x_0)^{\alpha}},$$

where $\Delta^{\alpha}(h(x) - h(x_0)) \cong \Gamma(\alpha+1)(h(x) - h(x_0))$.

If $h^{(k+1)\alpha}(x) = \overbrace{D^{\alpha}D^{\alpha}\ldots D^{\alpha}}^{(k+1)\text{ times}}h(x)$ exists for any $x \in I \subseteq \mathbb{R}$, then we say that $h \in D_{(k+1)\alpha}(I)$, where $k = 0, 1, 2, 3, \ldots$

Definition 3 ([23])**.** *Consider $h(x) \in C_{\alpha}[a,b]$. The local fractional integral is therefore defined as*

$${}_aI_b^{\alpha}h(x) = \frac{1}{\Gamma(\alpha+1)}\int_a^b h(x)(dx)^{\alpha} = \frac{1}{\Gamma(\alpha+1)}\lim_{\Delta x \to 0}\sum_{j=0}^{N-1} h(x_j)(\Delta x_j)^{\alpha}$$

with $\Delta x_j = x_{j+1} - x_j$ and $\Delta x = \max\{\Delta x_1, \Delta x_2, \ldots, \Delta x_{N-1}\}$, where $[x_j, x_{j+1}]$, $j = 0, 1, \ldots, N-1$ and $a = x_0 < x_1 < \cdots < x_N = b$ is a partition of interval $[a,b]$.

Here, it follows that ${}_aI_b^{\alpha}h(x) = 0$ if $a = b$ and ${}_aI_b^{\alpha}h(x) = -{}_bI_a^{\alpha}h(x)$ if $a < b$. If for any $x \in [a,b]$, there exists ${}_aI_b^{\alpha}h(x)$, then we denoted by $h(x) \in I_x^{\alpha}[a,b]$.

Lemma 1 ([23])**.** ***Local fractional integration is anti-differentiation:*** *Assume $h(x) = k^{(\alpha)}(x) \in C_{\alpha}[a,b]$, so we have*

$${}_aI_b^{\alpha}h(x) = k(b) - k(a).$$

Local fractional integration by parts: *Assuming $h,k \in D_{\alpha}[a,b]$ and $h^{(\alpha)}(x)$, $k^{(\alpha)}(x) \in C_{\alpha}[a,b]$, we obtain*

$${}_aI_b^{\alpha}h(x)k^{(\alpha)}(x) = h(x)k(x)|_a^b - {}_aI_b^{\alpha}h^{(\alpha)}(x)k(x).$$

Lemma 2 ([23])**.** *For $h(x) = x^{k\alpha}$, we have for all , $k \in \mathbb{R}$*

$$\frac{d^{\alpha}h(x)}{dx^{\alpha}} = \frac{\Gamma(1+k\alpha)}{\Gamma(1+(k-1)\alpha)} x^{(k-1)\alpha},$$

$$\frac{1}{\Gamma(1+\alpha)} \int_a^b h(x)(dx)^{\alpha} = \frac{\Gamma(1+k\alpha)}{\Gamma(1+(k+1)\alpha)} \left(b^{(k+1)\alpha} - a^{(k+1)\alpha} \right).$$

Lemma 3 (Generalized Hölder's inequality [5])**.** *Let $h, k \in C_{\alpha}[a,b]$, $p, q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, then*

$$\frac{1}{\Gamma(1+\alpha)} \int_a^b |h(x)k(x)|(dx)^{\alpha}$$
$$\leq \left(\frac{1}{\Gamma(1+\alpha)} \int_a^b |h(x)|^p (dx)^{\alpha} \right)^{\frac{1}{p}} \left(\frac{1}{\Gamma(1+\alpha)} \int_a^b |k(x)|^q (dx)^{\alpha} \right)^{\frac{1}{q}}.$$

Definition 4 (Generalized convex function [23])**.** *Let $h : I \subseteq \mathbb{R} \to \mathbb{R}^{\alpha}$. For any $x_1, x_2 \in I$ and $\lambda \in [0,1]$, if*

$$h(\lambda x_1 + (1-\lambda)x_2) \leq \lambda^{\alpha} h(x_1) + (1-\lambda)^{\alpha} h(x_2)$$

holds, then h is a generalized convex function on I.

The following are two simple examples of generalized convex functions:

1. $h(x) = x^{\alpha p}, x \geq 0, p > 1$.
2. $h(x) = E_{\alpha}(x^{\alpha})$, $x \in R$ where $E_{\alpha}(x^{\alpha}) = \sum\limits_{k=0}^{\infty} \frac{x^{\alpha k}}{\Gamma(1+k\alpha)}$ denotes the Mittag–Leffler function.

3. Main Results

In order to prove our results, we need the following lemma

Lemma 4. *Let $h : I \to \mathbb{R}^{\alpha}$ be a differentiable function on I°, $a, b \in I^{\circ}$ with $a < b$, and $h^{(\alpha)} \in C_{\alpha}[a,b]$, then the following equality holds*

$$\frac{1}{(15)^{\alpha}} \left(8^{\alpha} h\left(\frac{3a+b}{4}\right) - h\left(\frac{a+b}{2}\right) + 8^{\alpha} h\left(\frac{a+3b}{4}\right) \right) - \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}} \cdot {}_a I_b^{\alpha} h(x)$$
$$= \frac{(b-a)^{\alpha}}{(16)^{\alpha}} \left(\frac{1}{\Gamma(\alpha+1)} \int_0^1 x^{\alpha} h^{(\alpha)}\left((1-x)a + x\frac{3a+b}{4}\right)(dx)^{\alpha} \right.$$
$$+ \frac{1}{\Gamma(\alpha+1)} \int_0^1 \left(x - \frac{17}{15}\right)^{\alpha} h^{(\alpha)}\left((1-x)\frac{3a+b}{4} + x\frac{a+b}{2}\right)(dx)^{\alpha}$$
$$+ \frac{1}{\Gamma(\alpha+1)} \int_0^1 \left(x + \frac{2}{15}\right)^{\alpha} h^{(\alpha)}\left((1-x)\frac{a+b}{2} + x\frac{a+3b}{4}\right)(dx)^{\alpha}$$
$$\left. + \frac{1}{\Gamma(\alpha+1)} \int_0^1 (x-1)^{\alpha} h^{(\alpha)}\left((1-x)\frac{a+3b}{4} + xb\right)(dx)^{\alpha} \right).$$

Proof. Let

$$I_1 = \frac{1}{\Gamma(\alpha+1)} \int_0^1 x^\alpha h^{(\alpha)} \left((1-x)a + x\frac{3a+b}{4} \right) (dx)^\alpha,$$

$$I_2 = \frac{1}{\Gamma(\alpha+1)} \int_0^1 \left(x - \frac{17}{15} \right)^\alpha h^{(\alpha)} \left((1-x)\frac{3a+b}{4} + x\frac{a+b}{2} \right) (dx)^\alpha,$$

$$I_3 = \frac{1}{\Gamma(\alpha+1)} \int_0^1 \left(x + \frac{2}{15} \right)^\alpha h^{(\alpha)} \left((1-x)\frac{a+b}{2} + x\frac{a+3b}{4} \right) (dx)^\alpha$$

and

$$I_4 = \frac{1}{\Gamma(\alpha+1)} \int_0^1 (x-1)^\alpha h^{(\alpha)} \left((1-x)\frac{a+3b}{4} + xb \right) (dx)^\alpha.$$

Using the local fractional integration by parts, we obtain

$$\begin{aligned} I_1 &= \frac{4^\alpha}{(b-a)^\alpha} x^\alpha h\left((1-x)a + x\frac{3a+b}{4} \right) \Big|_{x=0}^{x=1} - \frac{4^\alpha \Gamma(\alpha+1)}{(b-a)^\alpha} \int_0^1 h\left((1-x)a + x\frac{3a+b}{4} \right) (dx)^\alpha \\ &= \frac{4^\alpha}{(b-a)^\alpha} h\left(\frac{3a+b}{4} \right) - \frac{4^\alpha \Gamma(\alpha+1)}{(b-a)^\alpha} \int_0^1 h\left((1-x)a + x\frac{3a+b}{4} \right) (dx)^\alpha \\ &= \frac{4^\alpha}{(b-a)^\alpha} h\left(\frac{3a+b}{4} \right) - \frac{(16)^\alpha \Gamma(\alpha+1)}{(b-a)^{2\alpha}\Gamma(\alpha+1)} \int_a^{\frac{3a+b}{4}} h(u)(du)^\alpha. \end{aligned} \tag{1}$$

Similarly, we obtain

$$\begin{aligned} I_2 &= \frac{4^\alpha}{(b-a)^\alpha} \left(x - \frac{17}{15} \right)^\alpha h\left((1-x)\frac{3a+b}{4} + x\frac{a+b}{2} \right) \Big|_{x=0}^{x=1} \\ &\quad - \frac{4^\alpha \Gamma(\alpha+1)}{(b-a)^\alpha} \int_0^1 h\left((1-x)\frac{3a+b}{4} + x\frac{a+b}{2} \right) (dx)^\alpha \\ &= \frac{(-8)^\alpha}{(15)^\alpha (b-a)^\alpha} h\left(\frac{a+b}{2} \right) - \frac{(-68)^\alpha}{(15)^\alpha (b-a)^\alpha} h\left(\frac{3a+b}{4} \right) \\ &\quad - \frac{4^\alpha \Gamma(\alpha+1)}{(b-a)^\alpha} \int_0^1 h\left((1-x)\frac{3a+b}{4} + x\frac{a+b}{2} \right) (dx)^\alpha \\ &= - \frac{8^\alpha}{(15)^\alpha (b-a)^\alpha} h\left(\frac{a+b}{2} \right) + \frac{(68)^\alpha}{(15)^\alpha (b-a)^\alpha} h\left(\frac{3a+b}{4} \right) \\ &\quad - \frac{(16)^\alpha \Gamma(\alpha+1)}{(b-a)^{2\alpha}\Gamma(\alpha+1)} \int_{\frac{3a+b}{4}}^{\frac{a+b}{2}} h(u)(du)^\alpha, \end{aligned} \tag{2}$$

$$\begin{aligned} I_3 &= \frac{4^\alpha}{(b-a)^\alpha} \left(x + \frac{2}{15} \right)^\alpha h\left((1-x)\frac{a+b}{2} + x\frac{a+3b}{4} \right) \Big|_{x=0}^{x=1} \\ &\quad - \frac{4^\alpha \Gamma(\alpha+1)}{(b-a)^\alpha} \int_0^1 h\left((1-x)\frac{a+b}{2} + x\frac{a+3b}{4} \right) (dx)^\alpha \end{aligned}$$

$$
\begin{aligned}
&= \frac{(68)^{\alpha}}{(15)^{\alpha}(b-a)^{\alpha}} h\left(\frac{a+3b}{4}\right) - \frac{8^{\alpha}}{(15)^{\alpha}(b-a)^{\alpha}} h\left(\frac{a+b}{2}\right) \\
&\quad - \frac{4^{\alpha}\Gamma(\alpha+1)}{(b-a)^{\alpha}} \int_0^1 h\left((1-x)\frac{a+b}{2} + x\frac{a+3b}{4}\right)(dx)^{\alpha} \\
&= \frac{(68)^{\alpha}}{(15)^{\alpha}(b-a)^{\alpha}} h\left(\frac{a+3b}{4}\right) - \frac{8^{\alpha}}{(15)^{\alpha}(b-a)^{\alpha}} h\left(\frac{a+b}{2}\right) \\
&\quad - \frac{(16)^{\alpha}\Gamma(\alpha+1)}{(b-a)^{2\alpha}\Gamma(\alpha+1)} \int_{\frac{a+b}{2}}^{\frac{a+3b}{4}} h(u)(du)^{\alpha}
\end{aligned} \tag{3}
$$

and

$$
\begin{aligned}
I_4 &= \frac{4^{\alpha}}{(b-a)^{\alpha}}(x-1)h\left((1-x)\frac{a+3b}{4} + xb\right)\Bigg|_{x=0}^{x=1} \\
&\quad - \frac{4^{\alpha}\Gamma(\alpha+1)}{(b-a)^{\alpha}} \int_0^1 h\left((1-x)\frac{a+3b}{4} + xb\right)(dx)^{\alpha} \\
&= \frac{4^{\alpha}}{(b-a)^{\alpha}} h\left(\frac{a+3b}{4}\right) - \frac{4^{\alpha}\Gamma(\alpha+1)}{(b-a)^{\alpha}} \int_0^1 h\left((1-x)\frac{a+3b}{4} + xb\right)(dx)^{\alpha} \\
&= \frac{4^{\alpha}}{(b-a)^{\alpha}} h\left(\frac{a+3b}{4}\right) - \frac{(16)^{\alpha}\Gamma(\alpha+1)}{(b-a)^{2\alpha}\Gamma(\alpha+1)} \int_{\frac{a+3b}{4}}^{b} h(u)(du)^{\alpha}.
\end{aligned} \tag{4}
$$

Summing (1)–(4), and then multiplying the resulting equality by $\left(\frac{b-a}{16}\right)^{\alpha}$, we obtain the desired result. □

Theorem 5. *Let $h : [a,b] \to \mathbb{R}^{\alpha}$ be a differentiable function on (a,b) such that $h \in D_{\alpha}[a,b]$ and $h^{(\alpha)} \in C_{\alpha}[a,b]$ with $0 \le a < b$. If $\left|h^{(\alpha)}\right|$ is generalized convex on $[a,b]$, then we have*

$$
\begin{aligned}
&\left| \frac{1}{(15)^{\alpha}} \left(8^{\alpha} h\left(\frac{3a+b}{4}\right) - h\left(\frac{a+b}{2}\right) + 8^{\alpha} h\left(\frac{a+3b}{4}\right)\right) - \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}\, {}_aI_b^{\alpha} h(x) \right| \\
&\le \frac{(b-a)^{\alpha}}{(16)^{\alpha}} \Bigg(\left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} - \frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right)\left(\left|h^{(\alpha)}(a)\right| + \left|h^{(\alpha)}(b)\right|\right) \\
&\quad + \left(\left(\frac{2}{15}\right)^{\alpha} \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} + 2^{\alpha}\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right)\left(\left|h^{(\alpha)}\left(\frac{3a+b}{4}\right)\right| + \left|h^{(\alpha)}\left(\frac{a+3b}{4}\right)\right|\right) \\
&\quad + \left(\left(\frac{34}{15}\right)^{\alpha} \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} - 2^{\alpha}\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right)\left|h^{(\alpha)}\left(\frac{a+b}{2}\right)\right| \Bigg).
\end{aligned}
$$

Proof. From Lemma 4, the properties of the modulus, and the generalized convexity of $\left|h^{(\alpha)}\right|$, we have

$$
\begin{aligned}
&\left| \frac{1}{(15)^{\alpha}} \left(8^{\alpha} h\left(\frac{3a+b}{4}\right) - h\left(\frac{a+b}{2}\right) + 8^{\alpha} h\left(\frac{a+3b}{4}\right)\right) - \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}\, {}_aI_b^{\alpha} h(x) \right| \\
&\le \frac{(b-a)^{\alpha}}{(16)^{\alpha}} \left(\frac{1}{\Gamma(\alpha+1)} \int_0^1 x^{\alpha} \left| h^{(\alpha)}\left((1-x)a + x\frac{3a+b}{4}\right) \right| (dx)^{\alpha} \right.
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\alpha+1)}\int_0^1 \left|x-\frac{17}{15}\right|^{\alpha}\left|h^{(\alpha)}\left((1-x)\frac{3a+b}{4}+x\frac{a+b}{2}\right)\right|(dx)^{\alpha}\\
&+\frac{1}{\Gamma(\alpha+1)}\int_0^1 \left|x+\frac{2}{15}\right|^{\alpha}\left|h^{(\alpha)}\left((1-x)\frac{a+b}{2}+x\frac{a+3b}{4}\right)\right|(dx)^{\alpha}\\
&+\frac{1}{\Gamma(\alpha+1)}\int_0^1 |x-1|^{\alpha}\left|h^{(\alpha)}\left((1-x)\frac{a+3b}{4}+xb\right)\right|(dx)^{\alpha}\Bigg)\\
\leq&\frac{(b-a)^{\alpha}}{(16)^{\alpha}}\left(\frac{1}{\Gamma(\alpha+1)}\int_0^1 x^{\alpha}\left((1-x)^{\alpha}\left|h^{(\alpha)}(a)\right|+x^{\alpha}\left|h^{(\alpha)}\left(\frac{3a+b}{4}\right)\right|\right)(dx)^{\alpha}\right.\\
&+\frac{1}{\Gamma(\alpha+1)}\int_0^1 \left(\frac{17}{15}-x\right)^{\alpha}\left((1-x)^{\alpha}\left|h^{(\alpha)}\left(\frac{3a+b}{4}\right)\right|+x^{\alpha}\left|h^{(\alpha)}\left(\frac{a+b}{2}\right)\right|\right)(dx)^{\alpha} \qquad (5)\\
&+\frac{1}{\Gamma(\alpha+1)}\int_0^1 \left(x+\frac{2}{15}\right)^{\alpha}\left((1-x)^{\alpha}\left|h^{(\alpha)}\left(\frac{a+b}{2}\right)\right|+x^{\alpha}\left|h^{(\alpha)}\left(\frac{a+3b}{4}\right)\right|\right)(dx)^{\alpha}\\
&\left.+\frac{1}{\Gamma(\alpha+1)}\int_0^1 (1-x)^{\alpha}\left((1-x)^{\alpha}\left|h^{(\alpha)}\left(\frac{a+3b}{4}\right)\right|+x^{\alpha}\left|h^{(\alpha)}(b)\right|\right)(dx)^{\alpha}\right)\\
=&\frac{(b-a)^{\alpha}}{(16)^{\alpha}\Gamma(\alpha+1)}\left(\left|h^{(\alpha)}(a)\right|\int_0^1 x^{\alpha}(1-x)^{\alpha}(dx)^{\alpha}\right.\\
&+\left|h^{(\alpha)}\left(\frac{3a+b}{4}\right)\right|\left(\int_0^1 x^{2\alpha}(dx)^{\alpha}+\int_0^1\left(\frac{17}{15}-x\right)^{\alpha}(1-x)^{\alpha}(dx)^{\alpha}\right)\\
&+\left|h^{(\alpha)}\left(\frac{a+b}{2}\right)\right|\left(\int_0^1\left(\frac{17}{15}-x\right)^{\alpha}x^{\alpha}(dx)^{\alpha}+\int_0^1\left(x+\frac{2}{15}\right)^{\alpha}(1-x)^{\alpha}(dx)^{\alpha}\right)\\
&+\left|h^{(\alpha)}\left(\frac{a+3b}{4}\right)\right|\left(\int_0^1\left(x+\frac{2}{15}\right)^{\alpha}x^{\alpha}(dx)^{\alpha}+\int_0^1(1-x)^{2\alpha}(dx)^{\alpha}\right)\\
&\left.+\int_0^1(1-x)^{\alpha}x^{\alpha}\left|h^{(\alpha)}(b)\right|(dx)^{\alpha}\right).
\end{aligned}
$$

Using a change of variable and Lemma 2, we easily find

$$\frac{1}{\Gamma(\alpha+1)}\int_0^1 x^{\alpha}(1-x)^{\alpha}(dx)^{\alpha}=\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}-\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}, \tag{6}$$

$$\frac{1}{\Gamma(\alpha+1)}\int_0^1 x^{2\alpha}(dx)^{\alpha}=\int_0^1(1-x)^{2\alpha}(dx)^{\alpha}=\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}, \tag{7}$$

$$
\begin{aligned}
\frac{1}{\Gamma(\alpha+1)}\int_0^1\left(\frac{17}{15}-x\right)^{\alpha}(1-x)^{\alpha}(dx)^{\alpha}=&\frac{1}{\Gamma(\alpha+1)}\int_0^1\left(x+\frac{2}{15}\right)^{\alpha}x^{\alpha}(dx)^{\alpha}\\
=&\left(\frac{2}{15}\right)^{\alpha}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}+\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)} \qquad (8)
\end{aligned}
$$

and

$$\frac{1}{\Gamma(\alpha+1)}\int_0^1 \left(\frac{17}{15}-x\right)^{\alpha} x^{\alpha}(dx)^{\alpha} = \frac{1}{\Gamma(\alpha+1)}\int_0^1 \left(x+\frac{2}{15}\right)^{\alpha}(1-x)^{\alpha}(dx)^{\alpha}$$
$$= \left(\frac{17}{15}\right)^{\alpha}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} - \frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}. \tag{9}$$

Using (6)–(9) in (5), we obtain

$$\left|\frac{1}{(15)^{\alpha}}\left(8^{\alpha}h\left(\frac{3a+b}{4}\right) - h\left(\frac{a+b}{2}\right) + 8^{\alpha}h\left(\frac{a+3b}{4}\right)\right) - \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}\,{}_aI_b^{\alpha}h(x)\right|$$
$$\leq \frac{(b-a)^{\alpha}}{(16)^{\alpha}}\left(\left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} - \frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right)\left(\left|h^{(\alpha)}(a)\right| + \left|h^{(\alpha)}(b)\right|\right)\right.$$
$$+ \left(\left(\frac{2}{15}\right)^{\alpha}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} + 2^{\alpha}\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right)\left(\left|h^{(\alpha)}\left(\frac{3a+b}{4}\right)\right| + \left|h^{(\alpha)}\left(\frac{a+3b}{4}\right)\right|\right)$$
$$\left. + \left(\left(\frac{34}{15}\right)^{\alpha}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} - 2^{\alpha}\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right)\left|h^{(\alpha)}\left(\frac{a+b}{2}\right)\right|\right),$$

which is the result. The proof is completed. □

Theorem 6. *Let $h : I \to \mathbb{R}^{\alpha}$ be a differentiable function on I°, $a, b \in I^{\circ}$ with $a < b$, such that $h \in D_{\alpha}[a,b]$ and $h^{(\alpha)} \in C_{\alpha}[a,b]$. If $\left|h^{(\alpha)}\right|^{q}$ is a generalized convex on $[a,b]$, where $q > 1$ with $\frac{1}{p}+\frac{1}{q} = 1$, then we have*

$$\left|\frac{1}{(15)^{\alpha}}\left(8^{\alpha}h\left(\frac{3a+b}{4}\right) - h\left(\frac{a+b}{2}\right) + 8^{\alpha}h\left(\frac{a+3b}{4}\right)\right) - \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}\,{}_aI_b^{\alpha}h(x)\right|$$
$$\leq \frac{(b-a)^{\alpha}}{(16)^{\alpha}}\left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\right)^{\frac{1}{q}}\left(\frac{\Gamma(1+p\alpha)}{\Gamma(1+(p+1)\alpha)}\right)^{\frac{1}{p}}\left(\left(\left|h^{(\alpha)}(a)\right|^{q} + \left|h^{(\alpha)}\left(\frac{3a+b}{4}\right)\right|^{q}\right)^{\frac{1}{q}}\right.$$
$$+ \left(\frac{(17)^{p+1}-2^{p+1}}{(15)^{p+1}}\right)^{\alpha\frac{1}{p}}\left(\left|h^{(\alpha)}\left(\frac{3a+b}{4}\right)\right|^{q} + \left|h^{(\alpha)}\left(\frac{a+b}{2}\right)\right|^{q}\right)^{\frac{1}{q}}$$
$$+ \left(\frac{(17)^{p+1}-2^{p+1}}{(15)^{p+1}}\right)^{\alpha\frac{1}{p}}\left(\left|h^{(\alpha)}\left(\frac{a+b}{2}\right)\right|^{q} + \left|h^{(\alpha)}\left(\frac{a+3b}{4}\right)\right|^{q}\right)^{\frac{1}{q}}$$
$$\left. + \left(\left|h^{(\alpha)}\left(\frac{a+3b}{4}\right)\right|^{q} + \left|h^{(\alpha)}(b)\right|^{q}\right)^{\frac{1}{q}}\right).$$

Proof. From Lemma 4, the properties of modulus, generalized Hölder's inequality and generalized convexity of $\left|h^{(\alpha)}\right|^{q}$, we have

$$\left|\frac{1}{(15)^{\alpha}}\left(8^{\alpha}h\left(\frac{3a+b}{4}\right) - h\left(\frac{a+b}{2}\right) + 8^{\alpha}h\left(\frac{a+3b}{4}\right)\right) - \frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}\,{}_aI_b^{\alpha}h(x)\right|$$
$$\leq \frac{(b-a)^{\alpha}}{(16)^{\alpha}}\left(\left(\frac{1}{\Gamma(\alpha+1)}\int_0^1 x^{p\alpha}(dx)^{\alpha}\right)^{\frac{1}{p}}\left(\frac{1}{\Gamma(\alpha+1)}\int_0^1\left|h^{(\alpha)}\left((1-x)a + x\frac{3a+b}{4}\right)\right|^{q}(dx)^{\alpha}\right)^{\frac{1}{q}}\right.$$
$$+ \left(\frac{1}{\Gamma(\alpha+1)}\int_0^1\left(\frac{17}{15}-x\right)^{p\alpha}(dx)^{\alpha}\right)^{\frac{1}{p}}$$

$$\times\left(\frac{1}{\Gamma(\alpha+1)}\int_0^1\left|h^{(\alpha)}\left((1-x)\frac{3a+b}{4}+x\frac{a+b}{2}\right)\right|^q(dx)^\alpha\right)^{\frac{1}{q}}$$
$$+\left(\frac{1}{\Gamma(\alpha+1)}\int_0^1\left(x+\frac{2}{15}\right)^{p\alpha}(dx)^\alpha\right)^{\frac{1}{p}}$$
$$\times\left(\frac{1}{\Gamma(\alpha+1)}\int_0^1\left|h^{(\alpha)}\left((1-x)\frac{a+b}{2}+x\frac{a+3b}{4}\right)\right|^q(dx)^\alpha\right)^{\frac{1}{q}}$$
$$+\left(\frac{1}{\Gamma(\alpha+1)}\int_0^1(1-x)^{p\alpha}(dx)^\alpha\right)^{\frac{1}{p}}\left(\frac{1}{\Gamma(\alpha+1)}\int_0^1\left|h^{(\alpha)}\left((1-x)\frac{a+3b}{4}+xb\right)\right|^q(dx)^\alpha\right)^{\frac{1}{q}}\Bigg)$$
$$\leq\frac{(b-a)^\alpha}{(16)^\alpha}\Bigg(\left(\frac{1}{\Gamma(\alpha+1)}\int_0^1x^{p\alpha}(dx)^\alpha\right)^{\frac{1}{p}}$$
$$\times\left(\frac{1}{\Gamma(\alpha+1)}\int_0^1\left((1-x)^\alpha\left|h^{(\alpha)}(a)\right|^q+x^\alpha\left|h^{(\alpha)}\left(\frac{3a+b}{4}\right)\right|^q\right)(dx)^\alpha\right)^{\frac{1}{q}} \quad (10)$$
$$+\left(\frac{1}{\Gamma(\alpha+1)}\int_0^1\left(\frac{17}{15}-x\right)^{p\alpha}(dx)^\alpha\right)^{\frac{1}{p}}$$
$$\times\left(\frac{1}{\Gamma(\alpha+1)}\int_0^1\left((1-x)^\alpha\left|h^{(\alpha)}\left(\frac{3a+b}{4}\right)\right|^q+x^\alpha\left|h^{(\alpha)}\left(\frac{a+b}{2}\right)\right|^q\right)(dx)^\alpha\right)^{\frac{1}{q}}$$
$$+\left(\frac{1}{\Gamma(\alpha+1)}\int_0^1\left(x+\frac{2}{15}\right)^{p\alpha}(dx)^\alpha\right)^{\frac{1}{p}}$$
$$\times\left(\frac{1}{\Gamma(\alpha+1)}\int_0^1\left((1-x)^\alpha\left|h^{(\alpha)}\left(\frac{a+b}{2}\right)\right|^q+x^\alpha\left|h^{(\alpha)}\left(\frac{a+3b}{4}\right)\right|^q\right)(dx)^\alpha\right)^{\frac{1}{q}}$$
$$+\left(\frac{1}{\Gamma(\alpha+1)}\int_0^1(1-x)^{p\alpha}(dx)^\alpha\right)^{\frac{1}{p}}$$
$$\times\left(\frac{1}{\Gamma(\alpha+1)}\int_0^1\left((1-x)^\alpha\left|h^{(\alpha)}\left(\frac{a+3b}{4}\right)\right|^q+x^\alpha\left|h^{(\alpha)}(b)\right|^q\right)(dx)^\alpha\right)^{\frac{1}{q}}\Bigg).$$

Using a change of variable, we have

$$\frac{1}{\Gamma(\alpha+1)}\int_0^1x^{p\alpha}(dx)^\alpha=\frac{1}{\Gamma(\alpha+1)}\int_0^1(1-x)^{p\alpha}(dx)^\alpha \quad (11)$$

and

$$\frac{1}{\Gamma(\alpha+1)}\int_0^1\left(\frac{17}{15}-x\right)^{p\alpha}(dx)^\alpha=\frac{1}{\Gamma(\alpha+1)}\int_0^1\left(\frac{2}{15}+x\right)^{p\alpha}(dx)^\alpha. \quad (12)$$

Using Lemma 2, we obtain

$$\frac{1}{\Gamma(\alpha+1)}\int_0^1 x^{p\alpha}(dx)^\alpha = \frac{\Gamma(1+p\alpha)}{\Gamma(1+(p+1)\alpha)}, \tag{13}$$

$$\frac{1}{\Gamma(\alpha+1)}\int_0^1 (1-x)^\alpha(dx)^\alpha = \frac{1}{\Gamma(\alpha+1)}\int_0^1 x^\alpha(dx)^\alpha = \frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} \tag{14}$$

and

$$\begin{aligned}\frac{1}{\Gamma(\alpha+1)}\int_0^1\left(\frac{17}{15}-x\right)^{p\alpha}(dx)^\alpha =& \frac{1}{\Gamma(\alpha+1)}\int_{\frac{2}{15}}^{\frac{17}{15}} x^{p\alpha}(dx)^\alpha \\ =& \frac{\Gamma(1+p\alpha)}{\Gamma(1+(p+1)\alpha)}\left(\left(\frac{17}{15}\right)^{(p+1)\alpha}-\left(\frac{2}{15}\right)^{(p+1)\alpha}\right) \\ =& \frac{\Gamma(1+p\alpha)}{\Gamma(1+(p+1)\alpha)}\left(\frac{(17)^{p+1}-2^{p+1}}{(15)^{p+1}}\right)^\alpha.\end{aligned} \tag{15}$$

Substituting (11)–(15) in (10), we obtain

$$\begin{aligned}&\left|\frac{1}{(15)^\alpha}\left(8^\alpha h\left(\frac{3a+b}{4}\right)-h\left(\frac{a+b}{2}\right)+8^\alpha h\left(\frac{a+3b}{4}\right)\right)-\frac{\Gamma(\alpha+1)}{(b-a)^\alpha}{}_aI_b^\alpha h(x)\right| \\ \leq& \frac{(b-a)^\alpha}{(16)^\alpha}\left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\right)^{\frac{1}{q}}\left(\frac{\Gamma(1+p\alpha)}{\Gamma(1+(p+1)\alpha)}\right)^{\frac{1}{p}}\left(\left(\left|h^{(\alpha)}(a)\right|^q+\left|h^{(\alpha)}\left(\frac{3a+b}{4}\right)\right|^q\right)^{\frac{1}{q}}\right. \\ &+\left(\frac{(17)^{p+1}-2^{p+1}}{(15)^{p+1}}\right)^{\alpha\frac{1}{p}}\left(\left|h^{(\alpha)}\left(\frac{3a+b}{4}\right)\right|^q+\left|h^{(\alpha)}\left(\frac{a+b}{2}\right)\right|^q\right)^{\frac{1}{q}} \\ &+\left(\frac{(17)^{p+1}-2^{p+1}}{(15)^{p+1}}\right)^{\alpha\frac{1}{p}}\left(\left|h^{(\alpha)}\left(\frac{a+b}{2}\right)\right|^q+\left|h^{(\alpha)}\left(\frac{a+3b}{4}\right)\right|^q\right)^{\frac{1}{q}} \\ &\left.+\left(\left|h^{(\alpha)}\left(\frac{a+3b}{4}\right)\right|^q+\left|h^{(\alpha)}(b)\right|^q\right)^{\frac{1}{q}}\right),\end{aligned}$$

which is the result. The proof is completed. □

Theorem 7. *Let $h: I \to \mathbb{R}^\alpha$ be a differentiable function on I°, $a,b \in I^\circ$ with $a < b$, such that $h \in D_\alpha[a,b]$ and $h^{(\alpha)} \in C_\alpha[a,b]$. If $\left|h^{(\alpha)}\right|^q$ is a generalized convex on $[a,b]$, where $q > 1$, then we have*

$$\begin{aligned}&\left|\frac{1}{(15)^\alpha}\left(8^\alpha h\left(\frac{3a+b}{4}\right)-h\left(\frac{a+b}{2}\right)+8^\alpha h\left(\frac{a+3b}{4}\right)\right)-\frac{\Gamma(\alpha+1)}{(b-a)^\alpha}{}_aI_b^\alpha h(x)\right| \\ \leq& \frac{(b-a)^\alpha}{(16)^\alpha}\left(\left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\right)^{1-\frac{1}{q}}\right. \\ &\times\left(\left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}-\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right)\left|h^{(\alpha)}(a)\right|^q+\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\left|h^{(\alpha)}\left(\frac{3a+b}{4}\right)\right|^q\right)^{\frac{1}{q}} \\ &+\left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left(\left(\frac{17}{15}\right)^{2\alpha}-\left(\frac{2}{15}\right)^{2\alpha}\right)\right)^{1-\frac{1}{q}}\end{aligned}$$

$$\times\left(\left(\left(\left(\frac{2}{15}\right)^{\alpha}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}+\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right)\left|h^{(\alpha)}\left(\frac{3a+b}{4}\right)\right|^{q}\right.\right.$$
$$\left.+\left(\left(\frac{17}{15}\right)^{\alpha}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}-\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right)\left|h^{(\alpha)}\left(\frac{a+b}{2}\right)\right|^{q}\right)^{\frac{1}{q}}$$
$$+\left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left(\left(\frac{17}{15}\right)^{2\alpha}-\left(\frac{2}{15}\right)^{2\alpha}\right)\right)^{1-\frac{1}{q}}$$
$$\times\left(\left(\left(\frac{17}{15}\right)^{\alpha}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}-\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right)\left|h^{(\alpha)}\left(\frac{a+b}{2}\right)\right|^{q}\right.$$
$$\left.+\left(\left(\frac{2}{15}\right)^{\alpha}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}+\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right)\left|h^{(\alpha)}\left(\frac{a+3b}{4}\right)\right|^{q}\right)^{\frac{1}{q}}+\left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\right)^{1-\frac{1}{q}}$$
$$\left.\times\left(\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\left|h^{(\alpha)}\left(\frac{a+3b}{4}\right)\right|^{q}+\left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}-\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right)\left|h^{(\alpha)}(b)\right|^{q}\right)^{\frac{1}{q}}\right).$$

Proof. From Lemma 4, the properties of the modulus, generalized power mean inequality, and generalized convexity of $\left|h^{(\alpha)}\right|^{q}$, we have

$$\left|\frac{1}{(15)^{\alpha}}\left(8^{\alpha}h\left(\frac{3a+b}{4}\right)-h\left(\frac{a+b}{2}\right)+8^{\alpha}h\left(\frac{a+3b}{4}\right)\right)-\frac{\Gamma(\alpha+1)}{(b-a)^{\alpha}}\cdot{}_{a}I_{b}^{\alpha}h(x)\right|$$
$$\leq\frac{(b-a)^{\alpha}}{(16)^{\alpha}}\left(\left(\frac{1}{\Gamma(\alpha+1)}\int_{0}^{1}x^{\alpha}(dx)^{\alpha}\right)^{1-\frac{1}{q}}\right.$$
$$\times\left(\frac{1}{\Gamma(\alpha+1)}\int_{0}^{1}x^{\alpha}\left|h^{(\alpha)}\left((1-x)a+x\frac{3a+b}{4}\right)\right|^{q}(dx)^{\alpha}\right)^{\frac{1}{q}}$$
$$+\left(\frac{1}{\Gamma(\alpha+1)}\int_{0}^{1}\left(\frac{17}{15}-x\right)^{\alpha}(dx)^{\alpha}\right)^{1-\frac{1}{q}}$$
$$\times\left(\frac{1}{\Gamma(\alpha+1)}\int_{0}^{1}\left(\frac{17}{15}-x\right)^{\alpha}\left|h^{(\alpha)}\left((1-x)\frac{3a+b}{4}+x\frac{a+b}{2}\right)\right|^{q}(dx)^{\alpha}\right)^{\frac{1}{q}}$$
$$+\left(\frac{1}{\Gamma(\alpha+1)}\int_{0}^{1}\left(x+\frac{2}{15}\right)^{\alpha}(dx)^{\alpha}\right)^{1-\frac{1}{q}}$$
$$\times\left(\frac{1}{\Gamma(\alpha+1)}\int_{0}^{1}\left(x+\frac{2}{15}\right)^{\alpha}\left|h^{(\alpha)}\left((1-x)\frac{a+b}{2}+x\frac{a+3b}{4}\right)\right|^{q}(dx)^{\alpha}\right)^{\frac{1}{q}}$$
$$+\left(\frac{1}{\Gamma(\alpha+1)}\int_{0}^{1}(1-x)^{\alpha}(dx)^{\alpha}\right)^{1-\frac{1}{q}}$$
$$\left.\times\left(\frac{1}{\Gamma(\alpha+1)}\int_{0}^{1}(1-x)^{\alpha}\left|h^{(\alpha)}\left((1-x)\frac{a+3b}{4}+xb\right)\right|^{q}(dx)^{\alpha}\right)^{\frac{1}{q}}\right)$$
$$\leq\frac{(b-a)^{\alpha}}{(16)^{\alpha}}\left(\left(\frac{1}{\Gamma(\alpha+1)}\int_{0}^{1}x^{\alpha}(dx)^{\alpha}\right)^{1-\frac{1}{q}}\right.$$

$$\times\left(\frac{1}{\Gamma(\alpha+1)}\int_0^1 x^\alpha\left((1-x)^\alpha\left|h^{(\alpha)}(a)\right|^q+x^\alpha\left|h^{(\alpha)}\left(\frac{3a+b}{4}\right)\right|^q\right)(dx)^\alpha\right)^{\frac{1}{q}}$$
$$+\left(\frac{1}{\Gamma(\alpha+1)}\int_0^1\left(\frac{17}{15}-x\right)^\alpha(dx)^\alpha\right)^{1-\frac{1}{q}}$$
$$\times\left(\frac{1}{\Gamma(\alpha+1)}\int_0^1\left(\frac{17}{15}-x\right)^\alpha\left((1-x)^\alpha\left|h^{(\alpha)}\left(\frac{3a+b}{4}\right)\right|^q+x^\alpha\left|h^{(\alpha)}\left(\frac{a+b}{2}\right)\right|^q\right)(dx)^\alpha\right)^{\frac{1}{q}}$$
$$+\left(\frac{1}{\Gamma(\alpha+1)}\int_0^1\left(x+\frac{2}{15}\right)^\alpha(dx)^\alpha\right)^{1-\frac{1}{q}} \tag{16}$$
$$\times\left(\frac{1}{\Gamma(\alpha+1)}\int_0^1\left(x+\frac{2}{15}\right)^\alpha\left((1-x)^\alpha\left|h^{(\alpha)}\left(\frac{a+b}{2}\right)\right|^q+x^\alpha\left|h^{(\alpha)}\left(\frac{a+3b}{4}\right)\right|^q\right)(dx)^\alpha\right)^{\frac{1}{q}}$$
$$+\left(\frac{1}{\Gamma(\alpha+1)}\int_0^1(1-x)^\alpha(dx)^\alpha\right)^{1-\frac{1}{q}}$$
$$\times\left(\frac{1}{\Gamma(\alpha+1)}\int_0^1(1-x)^\alpha\left((1-x)^\alpha\left|h^{(\alpha)}\left(\frac{a+3b}{4}\right)\right|^q+x^\alpha\left|h^{(\alpha)}(b)\right|^q\right)(dx)^\alpha\right)^{\frac{1}{q}}\Bigg).$$

Using a change of variable and Lemma 2, we have

$$\frac{1}{\Gamma(\alpha+1)}\int_0^1 x^\alpha(dx)^\alpha=\frac{1}{\Gamma(\alpha+1)}\int_0^1(1-x)^\alpha(dx)^\alpha=\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)} \tag{17}$$

and

$$\begin{aligned}\frac{1}{\Gamma(\alpha+1)}\int_0^1\left(\frac{17}{15}-x\right)^\alpha(dx)^\alpha&=\frac{1}{\Gamma(\alpha+1)}\int_0^1\left(\frac{2}{15}+x\right)^\alpha(dx)^\alpha\\&=\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left(\left(\frac{17}{15}\right)^{2\alpha}-\left(\frac{2}{15}\right)^{2\alpha}\right).\end{aligned} \tag{18}$$

Substituting (6)–(9), (17), and (18) in (16), we obtain

$$\begin{aligned}&\left|\frac{1}{(15)^\alpha}\left(8^\alpha h\left(\frac{3a+b}{4}\right)-h\left(\frac{a+b}{2}\right)+8^\alpha h\left(\frac{a+3b}{4}\right)\right)-\frac{\Gamma(\alpha+1)}{(b-a)^\alpha}{}_aI_b^\alpha h(x)\right|\\
\leq&\frac{(b-a)^\alpha}{(16)^\alpha}\left(\left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\right)^{1-\frac{1}{q}}\right.\\
&\times\left(\left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}-\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right)\left|h^{(\alpha)}(a)\right|^q+\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\left|h^{(\alpha)}\left(\frac{3a+b}{4}\right)\right|^q\right)^{\frac{1}{q}}\\
&+\left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left(\left(\frac{17}{15}\right)^{2\alpha}-\left(\frac{2}{15}\right)^{2\alpha}\right)\right)^{1-\frac{1}{q}}\\
&\times\left(\left(\left(\frac{2}{15}\right)^\alpha\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}+\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right)\left|h^{(\alpha)}\left(\frac{3a+b}{4}\right)\right|^q\right.\end{aligned}$$

$$
\begin{aligned}
&+\left(\left(\left(\frac{17}{15}\right)^{\alpha}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}-\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right)\left|h^{(\alpha)}\left(\frac{a+b}{2}\right)\right|^{q}\right)^{\frac{1}{q}}\\
&+\left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\left(\left(\frac{17}{15}\right)^{2\alpha}-\left(\frac{2}{15}\right)^{2\alpha}\right)\right)^{1-\frac{1}{q}}\\
&\times\left(\left(\left(\left(\frac{17}{15}\right)^{\alpha}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}-\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right)\left|h^{(\alpha)}\left(\frac{a+b}{2}\right)\right|^{q}\right.\right.\\
&+\left.\left(\left(\frac{2}{15}\right)^{\alpha}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}+\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right)\left|h^{(\alpha)}\left(\frac{a+3b}{4}\right)\right|^{q}(dt)^{\alpha}\right)^{\frac{1}{q}}\\
&+\left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\right)^{1-\frac{1}{q}}\\
&\times\left.\left(\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\left|h^{(\alpha)}\left(\frac{a+3b}{4}\right)\right|^{q}+\left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}-\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right)\left|h^{(\alpha)}(b)\right|^{q}\right)^{\frac{1}{q}}\right),
\end{aligned}
$$

which is the result. The proof is completed. □

4. Applications

Corrected dual-Simpson quadrature formula

Let Y be the partition of the points $a = x_0 < x_1 < \cdots < x_n = b$ of the interval $[a, b]$, and consider the quadrature formula

$$
\frac{1}{\Gamma(\alpha+1)}\int_a^b h(x)(dx)^{\alpha} = \lambda(h, \mathrm{Y}) + R(h, \mathrm{Y}),
$$

where

$$
\begin{aligned}
&\lambda(h,\mathrm{Y})\\
=&\frac{1}{\Gamma(\alpha+1)}\sum_{i=0}^{n-1}\left[\frac{(x_{i+1}-x_i)^{\alpha}}{(15)^{\alpha}}\left(8^{\alpha}h\left(\frac{3x_i+x_{i+1}}{4}\right)-h\left(\frac{x_i+x_{i+1}}{2}\right)+8^{\alpha}h\left(\frac{x_i+3x_{i+1}}{4}\right)\right)\right]
\end{aligned}
$$

and $R(h, \mathrm{Y})$ denotes the associated approximation error.

Proposition 1. *Let $n \in \mathbb{N}$ and $h : [a, b] \to \mathbb{R}^{\alpha}$ be a differentiable function on (a, b) with $0 \le a < b$ and $h^{(\alpha)} \in C_{\alpha}[a, b]$. If $\left|h^{(\alpha)}\right|$ is generalized convex function, we have*

$$
\begin{aligned}
|R(h,\mathrm{Y})| \le&\frac{1}{\Gamma(1+\alpha)}\sum_{i=0}^{n-1}\frac{(x_{i+1}-x_i)^{2\alpha}}{(16)^{\alpha}}\\
&\times\left[\left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}-\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right)\left(\left|h^{(\alpha)}(x_i)\right|+\left|h^{(\alpha)}(x_{i+1})\right|\right)\right.\\
&+\left(\left(\frac{2}{15}\right)^{\alpha}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}+2^{\alpha}\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right)\\
&\times\left(\left|h^{(\alpha)}\left(\frac{3x_i+x_{i+1}}{4}\right)\right|+\left|h^{(\alpha)}\left(\frac{x_i+3x_{i+1}}{4}\right)\right|\right)\\
&+\left.\left(\left(\frac{34}{15}\right)^{\alpha}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}-2^{\alpha}\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right)\left|h^{(\alpha)}\left(\frac{x_i+x_{i+1}}{2}\right)\right|\right].
\end{aligned}
$$

Proof. Applying Theorem 5 on the subintervals $[x_i, x_{i+1}]$ $(i = 0, 1, \ldots, n-1)$ of the partition Y, we obtain

$$\begin{aligned}
&\left|\frac{1}{(15)^{\alpha}}\left(8^{\alpha}h\left(\frac{3x_i+x_{i+1}}{4}\right)-h\left(\frac{x_i+x_{i+1}}{2}\right)+8^{\alpha}h\left(\frac{x_i+3x_{i+1}}{4}\right)\right)\right.\\
&\left.-\frac{\Gamma(\alpha+1)}{(x_{i+1}-x_i)^{\alpha}}\cdot{}_{x_i}I^{\alpha}_{x_{i+1}}h(x)\right|\\
\leq&\frac{(x_{i+1}-x_i)^{\alpha}}{(16)^{\alpha}}\left[\left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}-\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right)\left(\left|h^{(\alpha)}(x_i)\right|+\left|h^{(\alpha)}(x_{i+1})\right|\right)\right.\\
&+\left(\left(\frac{2}{15}\right)^{\alpha}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}+2^{\alpha}\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right)\\
&\times\left(\left|h^{(\alpha)}\left(\frac{3x_i+x_{i+1}}{4}\right)\right|+\left|h^{(\alpha)}\left(\frac{x_i+3x_{i+1}}{4}\right)\right|\right)\\
&\left.+\left(\left(\frac{34}{15}\right)^{\alpha}\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}-2^{\alpha}\frac{\Gamma(1+2\alpha)}{\Gamma(1+3\alpha)}\right)\left|h^{(\alpha)}\left(\frac{x_i+x_{i+1}}{2}\right)\right|\right].
\end{aligned}$$

Multiplying both sides of the above inequality by $\frac{1}{\Gamma(1+\alpha)}(x_{i+1}-x_i)^{\alpha}$, and then summing the obtained inequalities for all $i = 0, 1, \ldots, n-1$ and using the triangular inequality, we obtain the desired result. □

Application to special means

For arbitrary real numbers a, b we have:

The generalized arithmetic mean: $A(a,b) = \dfrac{a^{\alpha}+b^{\alpha}}{2^{\alpha}}$.

The generalized p-Logarithmic mean:

$$L_p(a,b)=\left[\frac{\Gamma(1+p\alpha)}{\Gamma(1+(p+1)\alpha)}\left(\frac{b^{(p+1)\alpha}-a^{(p+1)\alpha}}{(b-a)^{\alpha}}\right)\right]^{\frac{1}{p}},\quad a,b\in\mathbb{R}, a\neq b \text{ and } p\in\mathbb{Z}\setminus\{-1,0\}.$$

Proposition 2. *Let $a, b \in \mathbb{R}$ with $0 < a < b$, and $n \geq 2$, then we have*

$$\begin{aligned}
&\left|(16)^{\alpha}A\left(\left(\frac{3a+b}{4}\right)^{n},\left(\frac{a+3b}{4}\right)^{n}\right)-A^{n}(a,b)-(15)^{\alpha}\Gamma(\alpha+1)L^{n}_{n}(a,b)\right|\\
\leq&\frac{(15)^{\alpha}(b-a)^{\alpha}}{(16)^{\alpha}}\left(\frac{\Gamma(1+\alpha)}{\Gamma(1+2\alpha)}\right)^{\frac{1}{q}}\left(\frac{\Gamma(1+p\alpha)}{\Gamma(1+(p+1)\alpha)}\right)^{\frac{1}{p}}\left(\frac{\Gamma(1+n\alpha)}{\Gamma(1+(n-1)\alpha)}\right)^{\frac{1}{q}}\\
&\times\left(\left(a^{(n-1)\alpha q}+\left(\frac{3a+b}{4}\right)^{(n-1)\alpha q}\right)^{\frac{1}{q}}\right.\\
&+\left(\frac{(17)^{p+1}-2^{p+1}}{(15)^{p+1}}\right)^{\alpha\frac{1}{p}}\left(\left(\frac{3a+b}{4}\right)^{(n-1)\alpha q}+\left(\frac{a+b}{2}\right)^{(n-1)\alpha q}\right)^{\frac{1}{q}}\\
&+\left(\frac{(17)^{p+1}-2^{p+1}}{(15)^{p+1}}\right)^{\alpha\frac{1}{p}}\left(\left(\frac{a+b}{2}\right)^{(n-1)\alpha q}+\left(\frac{a+3b}{4}\right)^{(n-1)\alpha q}\right)^{\frac{1}{q}}\\
&\left.+\left(\left(\frac{a+3b}{4}\right)^{(n-1)\alpha q}+b^{(n-1)\alpha q}\right)^{\frac{1}{q}}\right).
\end{aligned}$$

Proof. The assertion follows from Theorem 6, applied to the function $f(x) = x^{n\alpha}$ where $f : (0, +\infty) \to \mathbb{R}^{\alpha}$. □

5. Conclusions

In this paper, some inequalities of the corrected Simpson-type integral for generalized convex functions are derived from a new generalized identity.

Our findings were shown to be effective when applied to the error estimates of the quadrature formula and to special means.

The results can lead to additional research in this fascinating field and generalizations in other types of calculations, including multiplicative calculus and quantum calculus.

Author Contributions: Conceptualization, A.L., W.S. and B.M.; methodology, A.L., W.S. and A.I.; validation, B.M. and A.I.; formal analysis, A.I.; investigation, W.S. and A.I.; writing—original draft preparation, A.L. and B.M.; writing—review and editing, A.L., W.S. and A.I.; visualization, W.S. and A.I.; supervision, B.M.; project administration, A.L. and W.S. All authors have read and agreed to the published version of the manuscript.

Funding: The authors declared that no funding was received for this article.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The work of the first author was supported by the MESRS of Algeria (PRFU project N° A14N01EP230220230001).

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Ahmad, J.; Mohyud-Din, S.T. Solving wave and diffusion equations on Cantor sets. *Proc. Pak. Acad. Sci.* **2015**, *52*, 81–87.
2. Yang, X.-J.; Srivastava, H.M.; He, J.-H.; Baleanu, D. Cantor-type cylindrical-coordinate method for differential equations with local fractional derivatives. *Phys. Lett. A* **2013**, *377*, 1696–1700. [CrossRef]
3. Yang, X.-J.; Srivastava, H.M. An asymptotic perturbation solution for a linear oscillator of free damped vibrations in fractal medium described by local fractional derivatives. *Commun. Nonlinear Sci. Numer. Simul.* **2015**, *29*, 499–504. [CrossRef]
4. Yang, X.-J.; Machado, J.A.T.; Nieto, J.J. A new family of the local fractional PDEs. *Fund. Inform.* **2017**, *151*, 63–75. [CrossRef]
5. Chen, G.-S. Generalizations of Hölder's and some related integral inequalities on fractal space. *J. Funct. Spaces Appl.* **2013**, *2013*, 198405. [CrossRef]
6. Liu, Q.; Sun, W. A Hilbert-type fractal integral inequality and its applications. *J. Inequal. Appl.* **2017**, *2017*, 83. [CrossRef] [PubMed]
7. Sarikaya, M.Z.; Tunç, T.; Budak, H. On generalized some integral inequalities for local fractional integrals. *Appl. Math. Comput.* **2016**, *276*, 316–323. [CrossRef]
8. Erden, S.; Sarikaya, M.Z. Generalized Pompeiu type inequalities for local fractional integrals and its applications. *Appl. Math. Comput.* **2016**, *274*, 282–291. [CrossRef]
9. Abdeljawad, T.; Rashid, S.; Hammouch, Z.; Chu, Y.-M. Some new local fractional inequalities associated with generalized (s,m)-convex functions and applications. *Adv. Differ. Equ.* **2020**, *2020*, 406. [CrossRef]
10. Sarikaya, M.Z.; Budak, H.; Erden, S. On new inequalities of Simpson's type for generalized convex functions. *Korean J. Math.* **2019**, *27*, 279–295.
11. Iftikhar, S.; Kumam, P.; Erden, S. Newton's-type integral inequalities via local fractional integrals. *Fractals* **2020**, *28*, 2050037. [CrossRef]
12. Mo, H.X.; Sui, X. Hermite-Hadamard-type inequalities for generalized s-convex functions on real linear fractal set $\mathbb{R}^{\alpha}$ $(0 < \alpha < 1)$. *Math. Sci.* **2017**, *11*, 241–246.
13. Akkurt, A.; Sarikaya, M.Z.; Budak, H.; Yildirim, H. Generalized Ostrowski type integral inequalities involving generalized moments via local fractional integrals. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat.* **2017**, *3*, 797–807. [CrossRef]
14. Al Quarashi, M.; Rashid, S.; Khalid, A.; Karaca, Y.; Chu, Y.-M. New computations of Ostrowski type inequality pertaining to fractal style with applications. *Fractals* **2021**, *29* , 2140026. [CrossRef]
15. Sarikaya, M.Z.; Budak, H. Generalized Ostrowski type inequalities for local fractional integrals. *Proc. Amer. Math. Soc.* **2017**, *145*, 1527–1538. [CrossRef]
16. Khan, Z.A.; Rashid, S.; Ashraf, R.; Baleanu, D.; Chu, Y.-M. Generalized trapezium-type inequalities in the settings of fractal sets for functions having generalized convexity property. *Adv. Differ. Equ.* **2020**, *657*, 657. [CrossRef]
17. Du, T.; Wang, H.; Khan, M.A.; Zhang, Y. Certain integral inequalities considering generalized m-convexity on fractal sets and their applications. *Fractals* **2019**, *27*, 1950117. [CrossRef]
18. Luo, C.; Wang, H.; Du, T. Fejér-Hermite-Hadamard type inequalities involving generalized h-convexity on fractal sets and their applications. *Chaos Solitons Fractals* **2020**, *131*, 109547. [CrossRef]

19. Meftah, B.; Souahi, A.; Merad, M. Some local fractional Maclaurin type inequalities for generalized convex functions and their applications. *Chaos Solitons Fractals* **2022**, *162*, 112504. [CrossRef]
20. Kilicman, A.; Saleh, W. Notions of generalized s-convex functions on fractal sets. *J. Inequal. Appl.* **2015**, *2015*, 312. [CrossRef]
21. Set, E.; Sarikaya, M.Z.; Uygun, N. On new inequalities of Simpson's type for generalized quasi convex functions. *Adv. Inequal. Appl.* **2017**, *2017*, 3.
22. Franjić, I.; Pečarić, J. On corrected dual Euler–Simpson formulae. *Soochow J. Math.* **2006**, *32*, 575–587.
23. Yang, X.-J. *Advanced Local Fractional Calculus and Its Applications*; World Science Publisher: New York, NY, USA, 2012.
24. Yang, Y.-J.; Baleanu, D.; Yang, X.-J. Analysis of fractal wave equations by local fractional Fourier series method. *Adv. Math. Phys.* **2013**, *2013*, 632309. [CrossRef]