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# An Outlook on Hybrid Fractional Modeling of a Heat Controller with Multi-Valued Feedback Control

**Shorouk M. Al-Issa [1,2,*,†], Ahmed M. A. El-Sayed [3,†] and Hind H. G. Hashem [3,†]**

1 Department of Mathematics, Faculty of Arts and Sciences, International University of Beirut, Beirut 1107, Lebanon
2 Department of Mathematics, Faculty of Arts and Sciences, Lebanese International University, Saida 1600, Lebanon
3 Department of Mathematics, Faculty of Science, Alexandria University, Alexandria 21544, Egypt; amasayed@alexu.edu.eg (A.M.A.E.-S.); h.hashem@adj.aast.edu (H.H.G.H.)
* Correspondence: shorouk.alissa@liu.edu.lb
† These authors contributed equally to this work.

**Abstract:** In this study, we extend the investigations of fractional-order models of thermostats and guarantee the solvability of hybrid Caputo fractional models for heat controllers, satisfying some nonlocal hybrid multi-valued conditions with multi-valued feedback control, which involves the Chandrasekhar kernel, by using hybrid Dhage's fixed point theorem. A part of this study is dedicated to transforming this problem into an equivalent integral representation and then proving some existence results to achieve our aims. Furthermore, the continuous dependence of the unique solution on the control variable and on the set of selections will be discussed. Moreover, we provide an illustration to support our results.

**Keywords:** Caputo fractional derivative; heat controller model; multi-valued boundary value problems; multi-valued feedback control

**MSC:** 26A33; 34K45; 47G10

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## 1. Introduction

Hundreds of years ago, humans tried to find a tool or an instrument that could control heat exchange or temperature. A device that could control heat transfer or temperature. The object was to have an easier life, allowing technology to take care of this virtual task. The result was the thermostat, which is found in furnaces, air conditioners, refrigerators, cars, etc. Many scholars have discussed mathematical models for thermostats, for example, refs. [1–10].

In 1997 [2], two new mathematical models characterizing the dynamic behavior of motor vehicle thermostats were introduced; these models including delay-differential equations have been solved. Another modern mathematical model of the energetic behavior of indoor regulators found in an engine's cooling framework was presented, in addition to a calculation of numerical solutions [3].

Webb [4] introduced a mathematical treatment for thermostats in 2005. The first mathematical model for thermostat control was developed by Webb [4] in the form:

$$\begin{cases} \nu''(\tau) + h(\tau)\,\mathcal{G}(\tau, \nu(\tau)) = 0 \\ \nu'(0) = 0,\ \mathscr{b}\,\nu'(\tau) + \nu(\tau) = 0,\ \tau \in [0,1],\ \mathscr{b} > 0 \end{cases}$$

A model for a thermostat with a second-order nonlocal boundary value problem was established in 2012 by Webb [5]. The sensors act linearly; one gives feedback to a controller at one endpoint from a portion of the interval, and the other provides feedback to a controller at the other extremity. Numerous positive solutions and a few nonexistent solutions were established by the proof of certain useful characteristics.

Second-order differential equations for inclusion and fractional hybrid versions of thermostat models have also been published [11]. On this issue, hybrid boundary value criteria have also been taken into consideration [11]. The system's historical memory is protected by the Caputo–Fabrizio fractional-order derivative, which has been used to model and study the complications of childhood mumps-related hearing loss in [12].

To characterize the energetic behavior of a car indoor regulator, two modern models including delay-differential conditions with hysteresis were formulated in 1997 [2], and it was discovered that these two models were solvable. An entirely novel mathematical representation of the dynamic behavior of a thermostat installed in an engine cooling circuit was demonstrated, along with a calculation of numerical results [13].

Hybrid differential equations have received great attention [1,9,14,15]. Dhage and Lakshmikantham [14] initiated and presented a discussion of hybrid differential equations. A generalized version of the hybrid Dhage's fixed point results was used by Baleanu et al. [15].

Shen et al. [6] established a fractional order model for a thermostat using the same boundary conditions as in [5].

A further extension of the second-order differential equation of a thermostat model to a fractional hybrid equation with nonlocal hybrid conditions has been considered in [11]:

$$-{}^{c}\mathfrak{D}^{\gamma}\left(\frac{z(\mathfrak{r})}{h(\mathfrak{r},z(\mathfrak{r}))}\right)\in\Psi(\mathfrak{r},z(\mathfrak{r})),\quad \mathfrak{r}\in[0,1] \tag{1}$$

with the hybrid conditions

$$\begin{cases}\mathfrak{D}\left(\frac{z(\mathfrak{r})}{h(\mathfrak{r},z(\mathfrak{r}))}\right)\Big|_{\mathfrak{r}=0}=0,\\ \lambda_1\,{}^{c}\mathfrak{D}^{\gamma-1}\left(\frac{z(\mathfrak{r})}{h(\mathfrak{r},z(\mathfrak{r}))}\right)\Big|_{\mathfrak{r}=1}+\lambda_2\left(\frac{z(\mathfrak{r})}{h(\mathfrak{r},z(\mathfrak{r}))}\right)\Big|_{\mathfrak{r}=\eta}=0,\end{cases} \tag{2}$$

Some existence results have been investigated. Moreover, two examples are illustrated in [11].

Recently, the authors of [9] have established a model for thermostats involving hybrid integro-differential inclusions:

$$-\frac{d^2}{d\mathfrak{r}^2}\left(\frac{\nu(\mathfrak{r})}{h(\mathfrak{r},\nu(\mathfrak{r}))}\right)\in\int_0^1\frac{\mathfrak{r}}{\mathfrak{r}+\tau}\Phi\left(\tau,\int_0^1\frac{\tau}{\tau+\varrho}\,\psi(\varrho,\nu(\varrho))\,d\varrho\right)d\tau,\ \mathfrak{r}\in[0,1]$$

satisfying the hybrid nonlocal conditions:

$$\begin{cases}\mathfrak{D}\left(\frac{\nu(\mathfrak{r})}{h(\mathfrak{r},\nu(\mathfrak{r}))}\right)\big|_{\mathfrak{r}=0}=0,\\ \lambda\,{}^{c}\mathfrak{D}^{\gamma}\left(\frac{\nu(\mathfrak{r})}{h(\mathfrak{r},\nu(\mathfrak{r}))}\right)\big|_{\mathfrak{r}=\alpha}+\left(\frac{\nu(\mathfrak{r})}{h(\mathfrak{r},\nu(\mathfrak{r}))}\right)\big|_{\mathfrak{r}=\eta}=0,\quad \gamma\in(0,1],\ \alpha\in(0,1],\ \eta\in(0,1]\end{cases}$$

and proved some existence and continuous dependency results.

Inspired by the investigated mathematical models in [9,11], we establish some existence results for hybrid fractional modeling of thermostats.

$$-{}^{c}\mathfrak{D}^{\gamma}\left(\frac{\mu(\mathfrak{r})}{\mathcal{F}(\mathfrak{r},\mu(\mathfrak{r}))}\right)\in\int_0^{\mathfrak{r}}\frac{\tau}{\mathfrak{r}+\tau}\Phi\left(\tau,\nu(\tau),\int_0^1\frac{\tau}{\tau+\varrho}\,\psi(\varrho,\mu(\varrho))\,d\varrho\right)d\tau,\quad \mathfrak{r}\in I=[0,1] \tag{3}$$

with a nonlocal hybrid multi-valued boundary condition

$$\begin{cases}\mathfrak{D}\left(\frac{\mu(\mathfrak{r})}{\mathcal{F}(\mathfrak{r},\mu(\mathfrak{r}))}\right)\Big|_{\mathfrak{r}=0}=0,\\ \lambda_1\,{}^{c}\mathfrak{D}^{\gamma-1}\left(\frac{\mu(\mathfrak{r})}{\mathcal{F}(\mathfrak{r},\mu(\mathfrak{r}))}\right)\Big|_{\mathfrak{r}=1}+\lambda_2\left(\frac{\mu(\mathfrak{r})}{\mathcal{F}(\mathfrak{r},\mu(\mathfrak{r}))}\right)\Big|_{\mathfrak{r}=\eta}\in\Theta(\mathfrak{r},\mu(\mathfrak{r})),\end{cases} \tag{4}$$

and with a multi-valued control variable in the form of

$$\nu(\mathfrak{r}) \in \Omega\big(\mathfrak{r}, \nu(\mathfrak{r}), \mu(\mathfrak{r})\big), \tag{5}$$

where $\gamma > 0$ is a real number with $n-1 < \gamma < n$, and $\lambda_i$, $i = 1,2$ are positive real parameters, $\mathscr{D} = \frac{d}{dt}$, ${}^c\mathscr{D}^\gamma$ is the Caputo derivative of order $\gamma$, where $\Phi, \Omega : I \times \mathbb{R} \times \mathbb{R} \to P(\mathbb{R})$, $\Theta : I \times \mathbb{R} \to P(\mathbb{R})$ are multi-valued maps, $\psi : I \times \mathbb{R} \to \mathbb{R}$ is continuous, and $\mathscr{F} \in C(I \times \mathbb{R}, \mathbb{R} \setminus \{0\})$.

This study is the first attempt to discuss the solvability of the hybrid fractional model of thermostats (3) satisfying the nonlocal hybrid multi-valued condition (4) under multi-valued constraints (5) in $C(I, \mathbb{R})$. Furthermore, it will be established that the solution of this problem is unique and it depends continuously on the control variable (5) and on the set of selections $S_\Phi$. Finally, an example is presented to clarify our results.

To reach our goal, we need to investigate the single-valued problem that corresponds to the mentioned problem (3) and (4)

$$-{}^c\mathscr{D}^\gamma\left(\frac{\mu(\mathfrak{r})}{\mathscr{F}(\mathfrak{r}, \mu(\mathfrak{r}))}\right) = \int_0^{\mathfrak{r}} \frac{\tau}{\mathfrak{r}+\tau}\phi\left(\tau, \nu(\tau), \int_0^1 \frac{\tau}{\tau+\varrho}\, \psi(\tau, \mu(\varrho))\, d\varrho\right) d\tau, \quad \mathfrak{r} \in I \tag{6}$$

with nonlocal hybrid condition

$$\begin{cases} \mathscr{D}\left(\frac{\mu(\mathfrak{r})}{\mathscr{F}(\mathfrak{r},\mu(\mathfrak{r}))}\right)\Big|_{\mathfrak{r}=0} = 0, \\ \lambda_1\, {}^c\mathscr{D}^{\gamma-1}\left(\frac{\mu(\mathfrak{r})}{\mathscr{F}(\mathfrak{r},\mu(\mathfrak{r}))}\right)\Big|_{\mathfrak{r}=1} + \lambda_2\left(\frac{\mu(\mathfrak{r})}{\mathscr{F}(\mathfrak{r},\mu(\mathfrak{r}))}\right)\Big|_{\mathfrak{r}=\eta} = \theta(\mathfrak{r}, \mu(\mathfrak{r})), \end{cases} \tag{7}$$

and the control variable is provided as

$$\nu(\mathfrak{r}) = \varpi\big(\mathfrak{r}, \nu(\mathfrak{r}), \mu(\mathfrak{r})\big), \tag{8}$$

with $\phi \in S_\Phi$, $\theta \in S_\Theta$, and $\varpi \in S_\Omega$.

Our problem (3) and (4) involves Chandrasekhar's kernel; integral equations containing this kernel have been treated and discussed by many scholars in different classes and by various techniques due to their usage in numerous branches of research and engineering, including traffic theory, neutron transport theory, kinetic theory of gases, and radiative transfer theory (for examples, see [16,17]).

## 2. Single-Valued Problem

Consider the nonlocal problem (6) and (7) with feedback control (8), assuming the following:

($\mathscr{H}_1$) $\phi : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous in $\nu, \tau$ for every $\mathfrak{r} \in I$, and measurable for almost all $\mathfrak{r} \in I$ and $\forall\ \nu, \tau \in \mathbb{R}$. There are two integrable functions $m, k_1 : I \to I$ with

$$|\phi(\mathfrak{r}, \nu, \tau)| \le m(\mathfrak{r}) + k_1(\mathfrak{r})(|\nu| + |\tau|), \ \ \mathfrak{r} \in I$$

and

$$\int_0^1 \frac{1}{\tau+\varrho}\, |m(\varrho)|\, d\varrho \le m, \ \text{ and } \ \int_0^1 \frac{1}{\tau+\varrho}\, |k_1(\varrho)|\, d\varrho \le k.$$

($\mathscr{H}_2$) $\psi \in C(I \times \mathbb{R}, \mathbb{R})$, and there exists a continuous function $k_2 : [0,1] \times [0,1] \to \mathbb{R}$, and a non-decreasing continuous map $\chi : [0, \infty) \to (0, \infty)$, with

$$|\psi(\mathfrak{r}, \tau)| \le k_2(\mathfrak{r})\chi(\|\tau\|),$$

and

$$\int_0^1 \frac{1}{\tau+\varrho}\, |k_2(\varrho)|\, d\varrho \le k^*.$$

($\mathscr{H}_3$) Let $\theta : I \times \mathbb{R} \to \mathbb{R}$ be a Lipschitzian function, with

$$|\theta(\mathfrak{r}, \mu_1) - \theta(\mathfrak{r}, \mu_2)| \le k_3(\mathfrak{r})|\mu_1 - \mu_2|.$$

($\mathscr{H}_4$) $\mathscr{F} \in C(I \times \mathbb{R}, \mathbb{R}\backslash\{0\})$ and there is a continuous function $\omega : I \to I$, where

$$|\mathscr{F}(\mathfrak{r}, \mu_1) - \mathscr{F}(\mathfrak{r}, \mu_2)| \le \omega(\mathfrak{r})\ |\mu_1 - \mu_2|,$$

$\forall\ \ \mu_1, \mu_2 \in \mathbb{R}$ and $\mathfrak{r} \in I$.

($\mathscr{H}_5$) $\omega \in C(I \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, and there exists a measurable and bounded function $\delta : I \to \mathbb{R}$, which has norm $\|\delta\|$, with

$$|\varpi(\mathfrak{r}, \nu(\mathfrak{r}), \mu(\mathfrak{r}))| \le \delta(\mathfrak{r}),\ \ \mathfrak{r} \in I,$$

where $\delta = \max_{\mathfrak{r}\in I}\{\|\delta\|\}$.

($\mathscr{H}_6$) The real number $r$ is the positive root of

$$\begin{aligned} -K_3\|\omega\|\, r^2 + \big(1 - \big[m\ \|\omega\| + \|\omega\|\ \theta + k\ \big(k^*\chi(\|\vartheta\|) + \|\delta\|\big)\big]\ \Lambda\ + k_3 G\big]\big)\ r \\ -G\big(\tfrac{1}{\lambda_2}(\Theta\ +\ k_3 1 + m) + [m + k\, k^*\chi(\|\mu\|)]\ \Lambda\big)\ = 0, \end{aligned}$$

where $G = \sup_{\mathfrak{r}\in I} |\mathscr{F}(\mathfrak{r}, 0)|$, and

$$\Lambda = \frac{1}{\Gamma(\gamma+1)} + \frac{\lambda_1}{\lambda_2} + \frac{\lambda_2\ \eta^\gamma}{\Gamma(\gamma+1)}. \tag{9}$$

**Remark 1.** *Using assumptions ($\mathscr{H}_3$) and ($\mathscr{H}_4$), we have*

$$|\theta(\mathfrak{r}, \mu)| \le k_3(\mathfrak{r})|\mu| + \Theta, \quad \Theta = \sup_{\mathfrak{r}\in I} |\theta(\mathfrak{r}, 0)|.$$

*and*

$$|\mathscr{F}(\mathfrak{r}, \mu)| \le \omega|\mu(\mathfrak{r})| + G,\ \ \textit{with}\ \ G = \sup_{\mathfrak{r}\in I} |\mathscr{F}(\mathfrak{r}, 0)|.$$

**Lemma 1.** *$\mu \in C(I, \mathbb{R})$ is a solution of the hybrid differential equation*

$$^c\mathscr{D}^\gamma\Big(\frac{\mu(\mathfrak{r})}{\mathscr{F}(\mathfrak{r}, \mu(\mathfrak{r}))}\Big) + \int_0^{\mathfrak{r}} \frac{\mathfrak{r}}{\mathfrak{r}+\tau}\ \chi(\tau)\ d\tau = 0 \ \ \mathfrak{r} \in [0,1], \gamma \in (1,2], \tag{10}$$

*with the condition (7) and feedback control (8) if $\mu \in C(I, \mathbb{R})$ is a solution of the following equation:*

$$\begin{aligned} \mu(\mathfrak{r}) =& \mathscr{F}(\mathfrak{r}, \mu(\mathfrak{r}))\Big[ - \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\tau)^{\gamma-1}}{\Gamma(\gamma)} \int_0^{\tau} \frac{\tau}{\tau+\varrho}\ \chi(\varrho)\ d\varrho d\tau + \frac{\lambda_1}{\lambda_2} \int_0^1 \int_0^{\tau} \frac{\tau}{\tau+\varrho}\chi(\varrho)\ d\varrho\ d\tau \\ &+ \int_0^{\eta} \frac{(\eta-\tau)^{\gamma-1}}{\Gamma(\gamma)} \int_0^{\tau} \frac{\tau}{\tau+\varrho}\ \chi(\varrho)\ d\varrho\ d\tau\Big] - \frac{\mathscr{F}(\mathfrak{r}, \mu(\mathfrak{r}))\ \theta(\mathfrak{r}, \mu(\mathfrak{r}))}{\lambda_2}. \end{aligned} \tag{11}$$

**Proof.** Assume that $\mu_0$ is a solution of (10). Then, we find constants $\alpha_0, \alpha_1 \in \mathbb{R}$ that satisfy

$$\mu_0 = \mathscr{F}(\mathfrak{r}, \mu_0(\mathfrak{r}))\Big[ - \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\tau)^{\gamma-1}}{\Gamma(\gamma)} \int_0^{\tau} \frac{\tau}{\tau+\varrho}\ \chi(\varrho)\ d\varrho d\tau\ + \alpha_0\ +\ \alpha_1\ \mathfrak{r}\Big]. \tag{12}$$

Then,

$$\mathscr{D}\Big(\frac{\mu_0(\mathfrak{r})}{\mathscr{F}(\mathfrak{r}, \mu_0(\mathfrak{r}))}\Big) = - \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\tau)^{\gamma-2}}{\Gamma(\gamma-1)} \int_0^{\tau} \frac{\tau}{\tau+\varrho}\ \chi(\varrho)\ d\varrho d\tau\ +\ \alpha_1$$

and

$$ {}^{c}\mathscr{D}^{\gamma-1}\Big(\frac{\mu_0(\mathfrak{r})}{\mathscr{F}(\mathfrak{r},\mu_0(\mathfrak{r}))}\Big) = -\int_0^{\mathfrak{r}}\int_0^{\tau}\frac{\tau}{\tau+\varrho}\,\chi(\varrho)\,d\varrho d\tau \;+\; \alpha_1\,\frac{\mathfrak{r}^{2-\gamma}}{\Gamma(3-\gamma)}. $$

Therefore, $\alpha_1 = 0$ and

$$ \alpha_0 = \frac{\theta(\mathfrak{r},\mu_0(\mathfrak{r}))}{\lambda_2} + \frac{\lambda_1}{\lambda_2}\int_0^1\int_0^{\tau}\frac{\tau}{\tau+\varrho}\chi(\varrho)\,d\varrho\,d\tau + \int_0^{\eta}\frac{(\eta-\tau)^{\gamma-1}}{\Gamma(\gamma)}\int_0^{\tau}\frac{\tau}{\tau+\varrho}\,\chi(\varrho)\,d\varrho\,d\tau. $$

We get, by replacing the values $\alpha_0$ and $\alpha_1$ in (12),

$$ \begin{aligned} \mu_0(\mathfrak{r}) =& \mathscr{F}(\mathfrak{r},\mu_0(\mathfrak{r}))\Big[-\int_0^{\mathfrak{r}}\frac{(\mathfrak{r}-\tau)^{\gamma-1}}{\Gamma(\gamma)}\int_0^{\tau}\frac{\tau}{\tau+\varrho}\,\chi(\varrho)\,d\varrho d\tau + \frac{\lambda_1}{\lambda_2}\int_0^1\int_0^{\tau}\frac{\tau}{\tau+\varrho}\chi(\varrho)\,d\varrho\,d\tau \\ &+ \int_0^{\eta}\frac{(\eta-\tau)^{\gamma-1}}{\Gamma(\gamma)}\int_0^{\tau}\frac{\tau}{\tau+\varrho}\,\chi(\varrho)\,d\varrho\,d\tau\Big] - \frac{\mathscr{F}(\mathfrak{r},\mu_0(\mathfrak{r}))\,\theta(\mathfrak{r},\mu_0(\mathfrak{r}))}{\lambda_2} \end{aligned} $$

This indicates that for the fractional integral Equation (11), $\mu_0$ is the solution. Conversely, it is obvious that for the fractional hybrid problem (7) and (10), $\mu_0$ is a solution of (11). □

**Corollary 1.** *Let $\mu \in C(I,\mathbb{R})$ be a solution of problem (6) and (7) with feedback control (8). Then, it satisfies*

$$ \begin{aligned} \mu(\mathfrak{r}) =& \mathscr{F}(\mathfrak{r},\mu(\mathfrak{r}))\Big[-\int_0^{\mathfrak{r}}\frac{(\mathfrak{r}-\tau)^{\gamma-1}}{\Gamma(\gamma)}\int_0^{\tau}\frac{\tau}{\tau+\varrho}\phi\Big(\varrho,\nu(\varrho),\int_0^{\varrho}\frac{\varrho}{\varrho+\varsigma}\,\psi(\varsigma,\mu(\varsigma))\,d\varsigma\Big)\,d\varrho\,d\tau \\ &+\frac{\lambda_1}{\lambda_2}\int_0^1\int_0^{\tau}\frac{\tau}{\tau+\varrho}\phi\Big(\varrho,\nu(\varrho),\int_0^{\varrho}\frac{\varrho}{\varrho+\varsigma}\,\psi(\varsigma,\mu(\varsigma))\,d\varsigma\Big)\,d\varrho\,d\tau \\ &+\int_0^{\eta}\frac{(\eta-\tau)^{\gamma-1}}{\Gamma(\gamma)}\int_0^{\tau}\frac{\tau}{\tau+\varrho}\phi\Big(\varrho,\nu(\varrho),\int_0^{\varrho}\frac{\varrho}{\varrho+\varsigma}\,\psi(\varsigma,\mu(\varsigma))d\varsigma\Big)d\varrho d\tau\Big] \\ &-\frac{\mathscr{F}(\mathfrak{r},\mu(\mathfrak{r}))\theta(\mathfrak{r},\mu(\mathfrak{r}))}{\lambda_2}. \end{aligned} \tag{13} $$

**Proof.** From Lemma 1, we have

$$ \begin{aligned} \mu(\mathfrak{r}) =& \mathscr{F}(\mathfrak{r},\mu(\mathfrak{r}))\Big[-\int_0^{\mathfrak{r}}\frac{(\mathfrak{r}-\tau)^{\gamma-1}}{\Gamma(\gamma)}\int_0^{\tau}\frac{\tau}{\tau+\varrho}\,\chi(\varrho)\,d\varrho d\tau + \frac{\lambda_1}{\lambda_2}\int_0^1\int_0^{\tau}\frac{\tau}{\tau+\varrho}\chi(\varrho)\,d\varrho\,ds \\ &+ \int_0^{\eta}\frac{(\eta-\tau)^{\gamma-1}}{\Gamma(\gamma)}\int_0^{\tau}\frac{\tau}{\tau+\varrho}\,\chi(\varrho)\,d\varrho\,d\tau\Big] - \frac{\mathscr{F}(\mathfrak{r},\mu(\mathfrak{r}))\,\theta(\mathfrak{r},\mu(\mathfrak{r}))}{\lambda_2}. \end{aligned} $$

Now, for

$$ \chi(\varrho) \;= \phi\Big(\varrho,\nu(\varrho),\int_0^{\varrho}\frac{\varrho}{\varrho+\varsigma}\,\psi(\varsigma,\mu(\varsigma))\,d\varsigma\Big),\quad \varrho,\varsigma\in I $$

we obtain the result. □

*2.1. Existence of Solutions*

**Theorem 1.** *Let ($\mathscr{H}_1$)–($\mathscr{H}_6$) be verified. Therefore, a solution for (13) exists.*

**Proof.** Consider the ball $\mathfrak{V}_\epsilon(0) = \{\tau\in X\;:\;\|\tau\|_X \le \epsilon\}$.

Clearly, $\mathfrak{V}_\epsilon(0)$ is a closed, convex, and bounded subset of the Banach space $X$. Regard the operators $\mathscr{A}: X\to X,\ \ \mathscr{B}:\mathfrak{V}_\epsilon(0)\to X$ defined by:

$$ (\mathscr{A}\mu)(\mathfrak{r}) = \digamma(\mathfrak{r},\mu(\mathfrak{r})),\quad \mathfrak{r}\in I \tag{14} $$

$$(\mathscr{B}\mu)(\mathfrak{r}) = -\int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\tau)^{\gamma-1}}{\Gamma(\gamma)} \int_0^{\tau} \frac{\tau}{\tau+\varrho}\phi\left(\varrho, \nu(\varrho), \int_0^{\varrho} \frac{\varrho}{\varrho+\varsigma}\, \psi(\varsigma, \mu(\varsigma))\, d\varsigma\right) d\varrho\, d\tau \\ + \frac{\lambda_1}{\lambda_2} \int_0^1 \int_0^{\tau} \frac{\tau}{\tau+\varrho}\phi\left(\varrho, \nu(\varrho), \int_0^{\varrho} \frac{\varrho}{\varrho+\varsigma}\, \psi(\varsigma, \mu(\varsigma))\, d\varsigma\right) d\varrho\, d\tau \\ + \int_0^{\eta} \frac{(\eta-\tau)^{\gamma-1}}{\Gamma(\gamma)} \int_0^{\tau} \frac{\tau}{\tau+\varrho}\phi\left(\varrho, \nu(\varrho), \int_0^{\varrho} \frac{\varrho}{\varrho+\varsigma}\, \psi(\varsigma, \mu(\varsigma))\, d\varsigma\right) d\varrho\, d\tau, \tag{15}$$

and

$$(\mathscr{C}\mu)(\mathfrak{r}) = \frac{\mathscr{F}(\mathfrak{r}, \mu(\mathfrak{r}))\, \theta(\mathfrak{r}, \mu(\mathfrak{r}))}{\lambda_2}. \tag{16}$$

As a solution to problem (6) and (7) with feedback control (8) exists, it is evident that X satisfies the operator equation $\mathscr{A}\mu\mathscr{B}\mu + \mathscr{C}\mu = \mu$. By utilizing the presumptions of the theorem of three operators with Banach algebra, due to Dhage [18] and the problem (6) and (7) with (8), we show that such a solution exists.

In the beginning, we demonstrate that operators $\mathscr{A}$, $\mathscr{C}$ are Lipschitzian with a constant $\|\omega\|$ on normed space $X$. For evidence of this, take $\mu_1, \mu_2 \in X$, then

$$\begin{aligned} |(\mathscr{A}\mu_1)(\mathfrak{r}) - (\mathscr{A}\mu_2)(\mathfrak{r})| &= |\mathscr{F}(\mathfrak{r}, \mu_1(\mathfrak{r})) - \mathscr{F}(\mathfrak{r}, \mu_2(\mathfrak{r}))| \\ &\leq \omega(\mathfrak{r})\, |\mu_1(\mathfrak{r}) - \mu_2(\mathfrak{r})|, \end{aligned}$$

$\forall\ \mu_1,\ \mu_2 \in \mathfrak{V}_\epsilon(0)$, and $\mathfrak{r} \in I$, and then

$$\|\mathscr{A}\mu_1 - \mathscr{A}\mu_2\|_{\mathfrak{X}} \leq \|\omega\|\ \|\mu_1 - \mu_2\|_{\mathfrak{X}},$$

The operator $\mathscr{A}$ is then Lipschitzian on $\mathfrak{V}_\epsilon(0)$ with the constant $\|\omega\|$.

Similarly, we have $\forall\ \mu_1,\ \mu_2 \in \mathfrak{V}_\epsilon(0)$

$$\begin{aligned} & |(\mathscr{C}\mu_1)(\mathfrak{r}) - (\mathscr{C}\mu_2)(\mathfrak{r})| \\ = \ & \frac{1}{\lambda_2}|\mathscr{F}(\mathfrak{r}, \mu_1(\mathfrak{r}))\theta(\mathfrak{r}, \mu_1(\mathfrak{r})) - \mathscr{F}(\mathfrak{r}, \mu_2(\mathfrak{r}))\theta(\mathfrak{r}, \mu_2(\mathfrak{r}))| \\ \leq \ & \frac{1}{\lambda_2}\left(|\mathscr{F}(\mathfrak{r}, \mu_1(\mathfrak{r}))|\ |\theta(\mathfrak{r}, \mu_1(\mathfrak{r})) - \theta(\mathfrak{r}, \mu_2(\mathfrak{r}))| + |\mathscr{F}(\mathfrak{r}, \mu_1(\mathfrak{r})) - \mathscr{F}(\mathfrak{r}, \mu_2(\mathfrak{r}))||\theta(\mathfrak{r}, \mu_2(\mathfrak{r}))|\right) \\ \leq \ & \frac{1}{\lambda_2}\left([\|\omega\||\mu_1(\mathfrak{r})| + G]\ k_3\ |\mu_1(\mathfrak{r}) - \mu_2(\mathfrak{r})| + \omega(\mathfrak{r})\ |\mu_1(\mathfrak{r}) - \mu_2(\mathfrak{r})|[\Theta\ +\ k_3(\mathfrak{r})\|u_2\|]\right) \\ \leq \ & \frac{([\|\omega\|\|\mu_1\| + G]\ k_3 + \|\omega\|[\Theta\ +\ k_3\|u_2\|])}{\lambda_2}\ |\mu_1(\mathfrak{r}) - \mu_2(\mathfrak{r})|. \end{aligned}$$

When applying the supremum to $\mathfrak{r} \in I$, we obtain

$$\|\mathscr{C}\mu_1 - \mathscr{C}\mu_2\|_X \leq \frac{([\|\omega\|\|\mu_1\| + G]\ k_3 + \|\omega\|[\Theta\ +\ k_3\|u_2\|])}{\lambda_2}\ \|\mu_1 - \mu_2\|_X,$$

Then, $\mathscr{C}$ is a Lipschitz mapping on $X$ with the Lipschitz constant $\frac{([\|\omega\|\|\mu_1\|+G]\ k_3+\|\omega\|[\Theta\ +\ k_3\|u_2\|])}{\lambda_2}$.

In second step, the aim is to show that $\mathscr{B}$ is continuous and compact on $\mathfrak{V}_\epsilon(0)$ into $X$; thus, the continuity of $\mathscr{B}$ on $\mathfrak{V}_\epsilon(0)$ is demonstrated. Taking $\{\mu_n\}$ as the series converges to $\mu \in \mathfrak{V}_\epsilon(0)$ and considering that $\mathfrak{r} \in I$, the continuous functions $\phi(\mathfrak{r}, \mu(\mathfrak{r}))$ and $\psi(\mathfrak{r}, \mu(\mathfrak{r}))$ in $X$ give $\phi(\mathfrak{r}, \mu_n(\mathfrak{r})) \to \phi(\mathfrak{r}, \mu(\mathfrak{r}))$, and $\psi(\mathfrak{r}, \mu_n(\mathfrak{r})) \to \psi(\mathfrak{r}, \mu(\mathfrak{r}))$ (from $(\mathscr{H}_2)$) and $(\mathscr{H}_3)$), by the Lebesgue Dominated Convergence Theorem. Then, we have

$$\lim_{n\to\infty}(\mathscr{B}\mu_n)(\mathfrak{r}) = -\lim_{n\to\infty}\int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\tau)^{\gamma-1}}{\Gamma(\gamma)} \int_0^{\tau} \frac{\tau}{\tau+\varrho}\phi\left(\varrho,\nu(\varrho),\int_0^{\varrho} \frac{\varrho}{\varrho+\varsigma}\,\psi(\varsigma,\mu_n(\varsigma))\,d\varsigma\right) d\varrho\, d\tau$$
$$+\lim_{n\to\infty}\frac{\lambda_1}{\lambda_2}\int_0^1\int_0^{\tau} \frac{\tau}{\tau+\varrho}\phi\left(\varrho,\nu(\varrho),\int_0^{\varrho} \frac{\varrho}{\varrho+\varsigma}\,\psi(\varsigma,\mu_n(\varsigma))\,d\varsigma\right) d\varrho\, d\tau$$
$$+\lim_{n\to\infty}\int_0^{\eta} \frac{(\eta-\tau)^{\gamma-1}}{\Gamma(\gamma)} \int_0^{\tau} \frac{\tau}{\tau+\varrho}\phi\left(\varrho,\nu(\varrho),\int_0^{\varrho} \frac{\varrho}{\varrho+\varsigma}\,\psi(\varsigma,\mu_n(\varsigma))\,d\varsigma\right) d\varrho\, d\tau$$
$$= -\int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\tau)^{\gamma-1}}{\Gamma(\gamma)} \int_0^{\tau} \frac{\tau}{\tau+\varrho}\phi\left(\varrho,\nu(\varrho),\int_0^{\varrho} \frac{\varrho}{\varrho+\varsigma}\,\lim_{n\to\infty}\psi(\varsigma,\mu_n(\varsigma))\,d\varsigma\right) d\varrho\, d\tau$$
$$+\frac{\lambda_1}{\lambda_2}\int_0^1\int_0^{\tau} \frac{\tau}{\tau+\varrho}\phi\left(\varrho,\nu(\varrho),\int_0^{\varrho} \frac{\varrho}{\varrho+\varsigma}\,\lim_{n\to\infty}\psi(\varsigma,\mu_n(\varsigma))\,d\varsigma\right) d\varrho\, d\tau$$
$$+\int_0^{\eta} \frac{(\eta-\tau)^{\gamma-1}}{\Gamma(\gamma)} \int_0^{\tau} \frac{\tau}{\tau+\varrho}\phi\left(\varrho,\nu(\varrho),\int_0^{\varrho} \frac{\varrho}{\varrho+\varsigma}\,\lim_{n\to\infty}\psi(\varsigma,\mu_n(\varsigma))\,d\varsigma\right) d\varrho\, d\tau$$
$$= -\int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\tau)^{\gamma-1}}{\Gamma(\gamma)} \int_0^{\tau} \frac{\tau}{\tau+\varrho}\phi\left(\varrho,\nu(\varrho),\int_0^{\varrho} \frac{\varrho}{\varrho+\varsigma}\,\lim_{n\to\infty}\psi(\varsigma,\mu_n(\varsigma))\,d\varsigma\right) d\varrho\, d\tau$$
$$+\frac{\lambda_1}{\lambda_2}\int_0^1\int_0^{\tau} \frac{\tau}{\tau+\varrho}\phi\left(\varrho,\nu(\varrho),\int_0^{\varrho} \frac{\varrho}{\varrho+\varsigma}\,\psi(\varsigma,\mu(\varsigma))\,d\varsigma\right) d\varrho\, d\tau$$
$$+\int_0^{\eta} \frac{(\eta-\tau)^{\gamma-1}}{\Gamma(\gamma)} \int_0^{\tau} \frac{\tau}{\tau+\varrho}\phi\left(\varrho,\nu(\varrho),\int_0^{\varrho} \frac{\varrho}{\varrho+\varsigma}\,\psi(\varsigma,\mu(\varsigma))\,d\varsigma\right) d\varrho\, d\tau = (\mathscr{B}\mu)(\mathfrak{r}),$$

for all $\mathfrak{r} \in I$. Thus, $\mathscr{B}\mu_n \to \mathscr{B}\mu$ as $n \to \infty$ uniformly on $\mathbb{R}$; then, the operator $\mathscr{B}$ is continuous on $\mathfrak{V}_\epsilon(0)$.

Next, we demonstrate that the operator $\mathscr{B}$ is compact on $\mathfrak{V}_\epsilon(0)$. It is sufficient to prove that $\mathscr{B}(\mathfrak{V}_\epsilon(0))$ is a uniformly bounded and equicontinuous set in $\mathfrak{V}_\epsilon(0)$. Using $(\mathscr{H}_2)$ and $(\mathscr{H}_3)$, then

$$|(\mathscr{B}\mu)(\mathfrak{r})|$$
$$\le \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\tau)^{\gamma-1}}{\Gamma(\gamma)} \int_0^{\tau} \frac{\tau}{\tau+\varrho}\left[m(\varrho)+k_1(\varrho)\left(|\nu(\varrho)|+\int_0^{\varrho}\frac{\varrho}{\varrho+\varsigma}\,|\psi(\varsigma,\mu(\varsigma))|\,d\varsigma\right)\right] d\varrho\, d\tau$$
$$+\frac{\lambda_1}{\lambda_2}\int_0^1\int_0^{\tau} \frac{\tau}{\tau+\varrho}\left[m(\varrho)+k_1(\varrho)\left(|\nu(\varrho)|+\int_0^{\varrho}\frac{\varrho}{\varrho+\varsigma}\,|\psi(\varsigma,\mu(\varsigma))|\,d\varsigma\right)\right] d\varrho\, d\tau$$
$$+\int_0^{\eta} \frac{(\eta-\tau)^{\gamma-1}}{\Gamma(\gamma)} \int_0^{\tau} \frac{\tau}{\tau+\varrho}\left[m(\varrho)+k_1(\varrho)\left(|\nu(\varrho)|+\int_0^{\varrho}\frac{\varrho}{\varrho+\varsigma}\,|\psi(\varsigma,\mu(\varsigma))|\,d\varsigma\right)\right] d\varrho\, d\tau$$
$$\le \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\tau)^{\gamma-1}}{\Gamma(\gamma)} \int_0^{1} \frac{\tau}{\tau+\varrho}\Big[|m(\varrho)|+|k_1(\varrho)|\big(|\varpi(\mathfrak{r},\nu(\mathfrak{r}),\mu(\mathfrak{r}))|$$
$$+\int_0^1\frac{\varrho}{\varrho+\varsigma}|k_2(\varsigma)|\chi(\|\mu\|)d\varsigma\big)\Big]d\varrho d\tau$$
$$+\frac{\lambda_1}{\lambda_2}\int_0^1\int_0^{1} \frac{\tau}{\tau+\varrho}\Big[|m(\varrho)|+|k_1(\varrho)|\big(|\varpi(\mathfrak{r},\nu(\mathfrak{r}),\mu(\mathfrak{r}))|$$
$$+\int_0^1\frac{\varrho}{\varrho+\varsigma}|k_2(\varsigma)|\chi(\|\mu\|)d\varsigma\big)\Big]\,d\varrho\,d\tau$$
$$+\int_0^{\eta} \frac{(\eta-\tau)^{\gamma-1}}{\Gamma(\gamma)} \int_0^{1} \frac{\tau}{\tau+\varrho}\Big[|m(\varrho)|+|k_1(\varrho)|\big(|\delta(\mathfrak{r})|+\int_0^1\frac{\varrho}{\varrho+\varsigma}|k_2(\varsigma)|\chi(\|\mu\|)d\varsigma\big)\Big]d\varrho d\tau$$
$$\le\left[\frac{m}{\Gamma(\gamma+1)}+\frac{\lambda_1}{\lambda_2}\,m+\frac{\lambda_2\,m\,\eta^{\gamma}}{\Gamma(\gamma+1)}\right]+k\left[\frac{1}{\Gamma(\gamma+1)}+\frac{\lambda_1}{\lambda_2}+\frac{\eta^{\gamma}}{\Gamma(\gamma+1)}\right]\left(k^*\chi(\|\mu\|)+\|\delta\|\right)$$
$$\le [m+k\,(k^*\chi(\|\mu\|)+\|\delta\|)]\,\Lambda,$$

$\forall\, \mathfrak{r} \in I$ and $\mu \in \mathfrak{V}_\epsilon(0)$. As a result, $\|\mathscr{B}\tau\| \le [m+k\,(k^*\chi(\|\tau\|)+\|\delta\|)]\,\Lambda$, such that $\Lambda$ is given in (9). As a consequence, it could be concluded that the set $\mathscr{B}(\mathfrak{V}_\epsilon(0))$ in the normed space $X$ is uniformly bounded. Hence, the equicontinuity of $\mathscr{B}$ is investigated. To achieve our goal, assuming $\mathfrak{r}_1, \mathfrak{r}_2 \in I$ with $\mathfrak{r}_1 < \mathfrak{r}_2$, then

$$\begin{aligned}
& |(\mathscr{B}\mu)(\mathfrak{r}_2) - (\mathscr{B}\mu)(\mathfrak{r}_1)| \\
\leq \; & \int_0^{\mathfrak{r}_1} \left( \frac{(\mathfrak{r}_2 - \tau)^{\gamma-1}}{\Gamma(\gamma)} - \frac{(\mathfrak{r}_1 - \tau)^{\gamma-1}}{\Gamma(\gamma)} \right) \\
\times \; & \int_0^{\tau} \frac{\tau}{\tau + \varrho} \big[ m(\varrho) + k_1(\varrho) \big( |\varpi(\mathfrak{r}, \nu(\mathfrak{r}), \mu(\mathfrak{r}))| + \int_0^1 \frac{\varrho}{\varrho + \varsigma} |k_2(\varsigma)| \chi(\|\mu\|) d\varsigma \big) \big] \, d\varrho \, d\tau \\
+ \; & \int_{\mathfrak{r}_1}^{\mathfrak{r}_2} \frac{(\mathfrak{r}_2 - \tau)^{\gamma-1}}{\Gamma(\gamma)} \int_0^{\tau} \frac{\tau}{\tau + \varrho} \big[ m(\varrho) + k_1(\varrho) \big( |\varpi(\mathfrak{r}, \nu(\mathfrak{r}), \mu(\mathfrak{r}))| \\
+ \; & \int_0^1 \frac{\varrho}{\varrho + \varsigma} |k_2(\varsigma)| \chi(\|\mu\|) d\varsigma \big) \big] \, d\varrho \, d\tau \\
\leq \; & \int_0^{\mathfrak{r}_1} \left( \frac{(\mathfrak{r}_2 - \tau)^{\gamma-1}}{\Gamma(\gamma)} - \frac{(\mathfrak{r}_1 - \tau)^{\gamma-1}}{\Gamma(\gamma)} \right) \int_0^1 \frac{\tau}{\tau + \varrho} \big[ |m(\varrho)| + |k_1(\varrho)| \big( |\delta(\varrho)| \\
+ \; & \int_0^1 \frac{\varrho}{\varrho + \varsigma} |k_2(\varsigma)| \chi(\|\mu\|) d\varsigma \big) \big] \, d\varrho \, d\tau \\
+ \; & \int_{\mathfrak{r}_1}^{\mathfrak{r}_2} \frac{(\mathfrak{r}_2 - \tau)^{\gamma-1}}{\Gamma(\gamma)} \int_0^1 \frac{\tau}{\tau + \varrho} \big[ |m(\varrho)| + |k_1(\varrho)| \big( |\delta(\varrho)| + \int_0^1 \frac{\varrho}{\varrho + \varsigma} |k_2(\varsigma)| \chi(\|\mu\|) d\varsigma \big) \big] \, d\varrho \, d\tau \\
\leq \; & [m + k(\|\delta\| + k^* \chi(\|\mu\|))] \left[ \int_0^{t_1} \left( \frac{(\mathfrak{r}_2 - \tau)^{\gamma-1}}{\Gamma(\gamma)} - \frac{(\mathfrak{r}_1 - \tau)^{\gamma-1}}{\Gamma(\gamma)} \right) d\tau + \int_{\mathfrak{r}_1}^{\mathfrak{r}_2} \frac{(\mathfrak{r}_2 - \tau)^{\gamma-1}}{\Gamma(\gamma)} d\tau \right],
\end{aligned}$$

This does not depend on $\mu \in \mathfrak{V}_\epsilon(0)$. Then, $\forall \, \varepsilon > 0$, and we find $\rho > 0$, where

$$|\mathfrak{r}_2 - \mathfrak{r}_1| < \rho \implies |(\mathscr{B}\mu)(\mathfrak{r}_2) - (\mathscr{B}\mu)(\mathfrak{r}_1)| < \varepsilon,$$

$\forall \, \mathfrak{r}_1, \mathfrak{r}_2 \in I$ , $\mu \in \mathfrak{V}_\epsilon(0)$. Then, $\forall \, \varepsilon > 0$. This demonstrates $\mathscr{B}$ in $X$ is an equicontinuous set. The Arzela–Ascoli theorem states that $\mathscr{B}$ is compact because it is equicontinuous and uniformly bounded on set $X$. Consequently, the operator $\mathscr{B}$ on $\mathfrak{V}_\epsilon(0)$ is completely continuous.
However, by utilizing ($\mathscr{H}_3$), then

$$\mathscr{M}_0^* = \|\mathscr{B}(\mathfrak{V}_\epsilon(0))\|_X = [m + k \, (\|\delta\| + k^* \, \chi(\|\mu\|))] \, \Lambda.$$

Putting $L^* = \|\omega\|$, then we get $L^* \mathscr{M}_0^* < 1$. Consequently, the Dhage hybrid fixed point theorem's [19] presumptions hold, and if either condition (a) or (b) is valid, then Dhage's hybrid fixed point theorem [19] is justified. In order for $\mu = \mathscr{A}\mu\mathscr{B}\vartheta + \mathscr{C}\mu$, let $\mu \in X$ and $\nu \in S$ be random elements. Then, there is

$$\begin{aligned}
& |\mu(\mathfrak{r})| \\
& = |\mathscr{A}\mu(\mathfrak{r})| |\mathscr{B}\nu(\mathfrak{r})| + |\mathscr{C}\mu(\mathfrak{r})| \\
& \leq |\mathscr{F}(\mathfrak{r}, \mu(\mathfrak{r}))| \Bigg( - \int_0^{\mathfrak{r}} \frac{(\mathfrak{r} - \tau)^{\gamma-1}}{\Gamma(\gamma)} \int_0^{\tau} \frac{\tau}{\tau + \varrho} \left| \phi\left( \varrho, \nu(\varrho), \int_0^{\varrho} \frac{\varrho}{\varrho + \varsigma} \, \psi(\varsigma, \vartheta(\varsigma)) \, d\varsigma \right) \right| d\varrho \, d\tau \\
& + \frac{\lambda_1}{\lambda_2} \int_0^1 \int_0^{\tau} \frac{\tau}{\tau + \varrho} \phi\left( \varrho, \nu(\varrho), \int_0^{\varrho} \frac{\varrho}{\varrho + \varsigma} \, \psi(\varsigma, \mu(\varsigma)) \, d\varsigma \right) \Bigg| \, d\varrho \, d\tau \\
& + \int_0^{\eta} \frac{(\eta - \tau)^{\gamma-1}}{\Gamma(\gamma)} \int_0^{\tau} \frac{\tau}{\tau + \varrho} \left| \phi\left( \varrho, \nu(\varrho), \int_0^{\varrho} \frac{\varrho}{\varrho + \varsigma} \, \psi(\varsigma, \vartheta(\varsigma)) \, d\varsigma \right) \right| d\varrho \, d\tau \Bigg) \\
& + \frac{([\|\omega\| \|\mu\| + G][\Theta \; + \; k_3 \|\mu\|])}{\lambda_2}
\end{aligned}$$

$$
\begin{aligned}
&\leq |\mathcal{F}(\mathfrak{r},\mu_1(\mathfrak{r}))| \bigg( \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\tau)^{\gamma-1}}{\Gamma(\gamma)} \int_0^{\tau} \frac{\tau}{\tau+\varrho} \big[ m(\varrho) + k_1(\varrho) \big( |\varpi(\varrho,\nu(\varrho),\mu(\varrho))| \\
&+ \int_0^1 \frac{\varrho}{\varrho+\varsigma} |k_2(\varsigma)| \chi(\|\vartheta\|) d\varsigma \big) \big] \, d\varrho \, d\tau \\
&+ \int_0^{\eta} \frac{(\eta-\tau)^{\gamma-1}}{\Gamma(\gamma)} \int_0^1 \frac{\tau}{\tau+\varrho} \big[ |m(\varrho)| + |k_1(\varrho)| \big( |\varpi(\mathfrak{r},\nu(\mathfrak{r}),\mu(\mathfrak{r}))| + \int_0^1 \frac{\varrho}{\varrho+\varsigma} |k_2(\varsigma)| \chi(\|\vartheta\|) d\varsigma \big) \big] d\varrho d\tau \\
&+ \frac{\lambda_1}{\lambda_2} \int_0^1 \int_0^{\tau} \frac{\tau}{\tau+\varrho} \big[ m(\varrho) + k_1(\varrho) \big( |\varpi(\mathfrak{r},\nu(\mathfrak{r}),\mu(\mathfrak{r}))| + \int_0^1 \frac{\varrho}{\varrho+\varsigma} |k_2(\varsigma)| \chi(\|\vartheta\|) d\varsigma \big) \big] \, d\varrho \, d\tau \\
&+ \int_0^{\eta} \frac{(\eta-\tau)^{\gamma-1}}{\Gamma(\gamma)} \int_0^1 \frac{\tau}{\tau+\varrho} \big[ |m(\varrho)| + |k_1(\varrho)| \big( |\varpi(\mathfrak{r},\nu(\mathfrak{r}),\mu(\mathfrak{r}))| + \int_0^1 \frac{\varrho}{\varrho+\varsigma} |k_2(\varsigma)| \chi(\|\vartheta\|) d\varsigma \big) \big] d\varrho d\tau \bigg) \\
&+ \frac{([\|\omega\| \|\mu\| + G][\Theta \, + \, k_3 \|\mu\|])}{\lambda_2}.
\end{aligned}
$$

Taking supremum over $\mathfrak{r} \in I$, we have

$$
\begin{aligned}
\|\mu\| \quad &\leq \quad [\|\mu\| \, \|\omega\| + G] \, [m + k \, (k^* \chi(\|\vartheta\|) + \|\delta\|)] \, \Lambda \, + \frac{([\|\mu\| \|\omega\| + G][\Theta \, + \, k_3 \|\mu\|])}{\lambda_2} \\
&\leq \quad \epsilon.
\end{aligned}
$$

Therefore, all of the requirements of Dhage's hybrid fixed point theorem [19] are held. Therefore, $\mu = \mathcal{A}\mu\mathcal{B}\vartheta + \mathcal{C}\mu$ has a solution in $S$. Hence, problem (6) and (7) with feedback control (8) is solvable in $S$ on $I$. $\square$

### 2.2. Existence of the Unique Solution

With the aim of proving some uniqueness results of the problem (6) and (7) involving (8), assume that:
$(\mathcal{H}_1)^*$ Let $\phi : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a Lipschitzian mapping, where

$$
|\phi(\mathfrak{r},\nu_1,\mu_1) - \phi(\mathfrak{r},,\nu_2,\mu_2)| \leq k_1(\mathfrak{r})(|\nu_1 - \nu_2| + |\mu_1 - \mu_2|).
$$

From this assumption, we see that the assumption $(\mathcal{H}_1)$ is valid; then,

$$
|\phi(\mathfrak{r},\nu,\mu)| \leq k_1(\mathfrak{r})(|\nu| + |\mu|) + m, \qquad m = \sup_{\mathfrak{r}\in I} |\phi(\mathfrak{r},0,0)|.
$$

$(\mathcal{H}_2)^*$ $\psi(\mathfrak{r},\mu(\mathfrak{r})) \;=\; k_2(\mathfrak{r}) \; \mu(\mathfrak{r})$.

**Theorem 2.** *Let the assumptions of Theorem 1 hold, and replace assumption $(\mathcal{H}_1)$ with $(\mathcal{H}_1)^*$ and $(\mathcal{H}_2)$ with $(\mathcal{H}_2)^*$ with*

$$
\Lambda \big[ \|\omega\| (m + k_1(\|\delta\| + k_2 r)) + (\|\omega\| r + G) k_1 \, k_2 \big] + \frac{1}{\lambda_2} \big[ k_3 \big[ \|\omega\| r + G \big] + \|\omega\| \, \big[ k_3 r + \Theta \big] \big] < 1.
$$

*Then, the hybrid problem (6) and (7) with feedback control (8) has a unique continuous solution.*

**Proof.** Let $\mu_1, \; \mu_2$ be two solutions of Equation (11), so

$$
\begin{aligned}
&|\mu_1(\mathfrak{r})-\mu_2(\mathfrak{r})| \\
&\le |\mathcal{F}(\mathfrak{r},\mu_1(\mathfrak{r}))-\mathcal{F}(\mathfrak{r},\mu_2(\mathfrak{r}))| \\
&\times \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\tau)^{\gamma-1}}{\Gamma(\gamma)} \int_0^{\tau} \frac{\tau}{\tau+\varrho}\Big[m(\varrho)+k_1(\varrho)\big(|\varpi(\varrho,\nu(\varrho),\mu_1(\varrho))|+\int_0^{\varrho}\frac{\varrho}{\varrho+\varsigma}\,\big|\psi(\varsigma,\mu_1(\varsigma))\big|\,d\varsigma\big)\Big]\,d\varrho\,d\tau \\
&+|\mathcal{F}(\mathfrak{r},\mu_2(\mathfrak{r}))|\int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\tau)^{\gamma-1}}{\Gamma(\gamma)}\int_0^{\tau}\frac{\tau}{\tau+\varrho}k_1(\varrho)\int_0^{\varrho}\frac{\varrho}{\varrho+\varsigma}\,\big|\psi(\varsigma,\mu_1(\varsigma))-\psi(\varsigma,\mu_2(\varsigma))\big|d\varsigma d\varrho d\tau \\
&+\frac{\lambda_1}{\lambda_2}\,|\mathcal{F}(\mathfrak{r},\mu_1(\mathfrak{r}))-\mathcal{F}(\mathfrak{r},\mu_2(\mathfrak{r}))| \\
&\times \int_0^{1}\int_0^{\tau}\frac{\tau}{\tau+\varrho}\Big[m(\varrho)+k_1(\varrho)\big(|\varpi(\varrho,\nu(\varrho),\mu_1(\varrho))|+\int_0^{\varrho}\frac{\varrho}{\varrho+\varsigma}\,k_1(\varsigma)\,\big|\psi(\varsigma,\mu_1(\varsigma))\big|\,d\varsigma\big)\Big]\,d\varrho\,d\tau \\
&+\frac{\lambda_1}{\lambda_2}|\mathcal{F}(\mathfrak{r},\mu_2(\mathfrak{r}))|\int_0^{1}\int_0^{\tau}\frac{\tau}{\tau+\varrho}k_1(\varrho)\int_0^{\varrho}\frac{\varrho}{\varrho+\varsigma}\big|\psi(\varsigma,\mu_1(\varsigma))-\psi(\varsigma,\mu_2(\varsigma))\big|d\varsigma d\varrho d\tau \\
&+|\mathcal{F}(\mathfrak{r},\mu_1(\mathfrak{r}))-\mathcal{F}(\mathfrak{r},\mu_2(\mathfrak{r}))| \\
&\times \int_0^{\eta} \frac{(\eta-\tau)^{\gamma-1}}{\Gamma(\gamma)} \int_0^{\tau} \frac{\tau}{\tau+\varrho}\Big[m(\varrho)+k_1(\varrho)\big(|\varpi(\varrho,\nu(\varrho),\mu_1(\varrho))|+\int_0^{\varrho}\frac{\varrho}{\varrho+\varsigma}\,\big|\psi(\varsigma,\mu_1(\varsigma)))\big|\,d\varsigma\big)\Big]\,d\varrho\,d\tau \\
&+|\mathcal{F}(\mathfrak{r},\mu_2(\mathfrak{r}))|\int_0^{\eta} \frac{(\eta-\tau)^{\gamma-1}}{\Gamma(\gamma)}\int_0^{\tau}\frac{\tau}{\tau+\varrho}k_1(\varrho)\int_0^{\varrho}\frac{\varrho}{\varrho+\varsigma}\big|\psi(\varsigma,\mu_1(\varsigma))-\psi(\varsigma,\mu_2(\varsigma))\big|d\varsigma d\varrho d\tau \\
&+\frac{1}{\lambda_2}\Big[|\mathcal{F}(\mathfrak{r},\mu_2(\mathfrak{r}))|\;|\theta(\mathfrak{r},\mu_2(\mathfrak{r}))-\;\theta(\mathfrak{r},\mu_1(\mathfrak{r}))|+\;|\mathcal{F}(\mathfrak{r},\mu_2(\mathfrak{r}))-\;\mathcal{F}(\mathfrak{r},\mu_1(\mathfrak{r}))|\;|\theta(\mathfrak{r},\mu_1(\mathfrak{r}))|\Big] \\
&\le \frac{\|\omega\|\;\|\mu_1-\mu_2\|}{\Gamma(\gamma+1)}\big[m+k_1\big(\|\delta\|+\;k_2\|\mu_1\|\big)\big]+\frac{\big[\|\omega\|\;\|\mu_2\|+G\big]}{\Gamma(\gamma+1)}k_1\,k_2\,\|\mu_1-\mu_2\| \\
&+\frac{\lambda_1}{\lambda_2}\,\|\omega\|\;\|\mu_1-\mu_2\|\big[m+k_1\big(\|\delta\|+\;k_2\,\|\mu_1\|\big)\big] \\
&+\frac{\lambda_1}{\lambda_2}\big[\|\omega\|\;|\mu_2(\mathfrak{r})|+G\big]\;k_1\;\;k_2\|\mu_1-\mu_2\| \\
&+\frac{\eta^{\gamma}\;\|\omega\|\;\|\mu_1-\mu_2\|}{\Gamma(\gamma+1)}\big[m+k_1\;\big(\|\delta\|+\;k_2\;\|\mu_1\|\big)\big]+\frac{\eta^{\gamma}\;\big[\|\omega\|\;\|\mu_2\|+G\big]}{\Gamma(\gamma+1)}k_1\,k_2\,\|\mu_1-\mu_2\| \\
&+\frac{1}{\lambda_2}\big[\,k_3\;\;\|\mu_1-\mu_2\|\;\big[\|\omega\|\|\mu_2\|+G\big]+\;\|\omega\|\;\|\mu_1-\mu_2\|\;\big[k_3\|\mu_1\|+\Theta\big]\big].
\end{aligned}
$$

Taking the supremum over $\mathfrak{r}\in I$, we have

$$
\begin{aligned}
\|\mu_1-\mu_2\| \le\;& \Big[\frac{1}{\Gamma(\gamma+1)}+\frac{\lambda_1}{\lambda_2}+\frac{\eta^{\gamma}}{\Gamma(\gamma+1)}\Big]\;\|\omega\|\;\|\mu_1-\mu_2\|\big[m+k_1\;\big(\|\delta\|+\;k_2\,r\big)\big] \\
&+\;\Big[\frac{1}{\Gamma(\gamma+1)}+\frac{\lambda_1}{\lambda_2}+\frac{\eta^{\gamma}}{\Gamma(\gamma+1)}\Big]\;\big[\|\omega\|\;r+G\big]k_1\,k_2\|\mu_1-\mu_2\| \\
&+\|\mu_1-\mu_2\|\;\big[\,k_3\;\;\big[\|\omega\|\|\mu_2\|+G\big]+\;\|\omega\|\;\;\big[k_3\|\mu_1\|+\Theta\big]\big] \\
\le\;&\;\|\mu_1-\mu_2\|\bigg(\Lambda\;\big[\omega(m+k_1\;(\|\delta\|+\;k_2\,r))+(\|\omega\|\;r+G)k_1\,k_2\big] \\
&+\frac{1}{\lambda_2}\big[\,k_3\;\;\big[\|\omega\|\|\mu_2\|+G+\;\|\omega\|\;\;\big[k_3\|\mu_1\|+\Theta\big]\big]\bigg),
\end{aligned}
$$

and

$$
\begin{aligned}
&\big[1-\big(\Lambda\;\big[\|\omega\|(m+k_1\;(\|\delta\|+\;k_2\,r))+(\|\omega\|\;r+G)k_1\;\;k_2\big] \\
&+\frac{1}{\lambda_2}\big[\,k_3\;\;\big[\|\omega\|r+G\big]+\;\|\omega\|\;\;\big[k_3r+\Theta\big]\big]\big)\big]\|\mu_1-\mu_2\|\le 0,
\end{aligned}
$$

which denotes

$$\mu_1(\mathfrak{r}) = \mu_2(\mathfrak{r}).$$

□

*2.3. Continuous Dependency on the Control Variable*

Here, we study the continuous dependence on the control variable $\nu$.

**Definition 1.** *The solution of problem (6) and (7) with feedback control (8) depends continuously on $\nu$ if $\forall\, \epsilon > 0, \ \exists\, \Upsilon > 0$, where*

$$|\nu(\mathfrak{r}) - \nu^*(\mathfrak{r})| = |\varpi(\mathfrak{r}, \nu, \mu) - \varpi(\mathfrak{r}, \nu^*, \mu)| \quad < \quad \Upsilon, \quad \mathfrak{r} \in [0,1].$$

*Then, $\|\mu - \mu^*\| < \epsilon$.*

Now, we prove the result.

**Theorem 3.** *Suppose that Theorem 5 is verified. Thus, the solution of (13) depends continuously on the control variable $\nu$.*

**Proof.** For the two solutions $\mu$ and $\mu^*$ of (13), corresponding to the control variables $\nu, \nu^*$, we get

$$\begin{aligned}
&|\mu(\mathfrak{r}) - \mu^*(\mathfrak{r})| \\
&\leq |\mathscr{F}(\mathfrak{r},\mu(\mathfrak{r})) - \mathscr{F}(\mathfrak{r},\mu^*(\mathfrak{r}))| \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\tau)^{\gamma-1}}{\Gamma(\gamma)} \int_0^{\tau} \frac{\tau}{\tau+\varrho} \left|\phi\left(\varrho, \nu(\varrho), \int_0^{\varrho} \frac{\varrho}{\varrho+\varsigma}\psi(\varsigma,\mu(\varsigma))d\varsigma\right)\right| d\varrho d\tau \\
&+ |\mathscr{F}(\mathfrak{r},\mu^*(\mathfrak{r}))| \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\tau)^{\gamma-1}}{\Gamma(\gamma)} \int_0^{\tau} \frac{\tau}{\tau+\varrho} \left[|\phi\left(\varrho, \nu(\varrho), \int_0^{\varrho} \frac{\varrho}{\varrho+\varsigma}\psi(\varsigma,\mu(\varsigma))d\varsigma\right)\right. \\
&- \phi\left(\varrho, \nu(\varrho), \int_0^{\varrho} \frac{\varrho}{\varrho+\varsigma}\psi(\varsigma,\mu^*(\varsigma))d\varsigma\right)\Bigg| \\
&+ \left|\phi\left(\varrho, \nu(\varrho), \int_0^{\varrho} \frac{\varrho}{\varrho+\varsigma}\psi(\varsigma,\mu^*(\varsigma))d\varsigma\right) - \phi\left(\varrho, \nu^*(\varrho), \int_0^{\varrho} \frac{\varrho}{\varrho+\varsigma}\psi(\varsigma,\mu^*(\varsigma))d\varsigma\right)\right|\Bigg] d\varrho d\tau \\
&+ \frac{\lambda_1}{\lambda_2}\, |\mathscr{F}(\mathfrak{r},\mu(\mathfrak{r})) - \mathscr{F}(\mathfrak{r},\mu^*(\mathfrak{r}))| \int_0^1 \int_0^{\tau} \frac{\tau}{\tau+\varrho} \left|\phi\left(\varrho, \nu(\varrho), \int_0^{\varrho} \frac{\varrho}{\varrho+\varsigma}\, \psi(\varsigma,\mu(\varsigma))\, d\varsigma\right)\right| d\varrho d\tau \\
&+ \frac{\lambda_1}{\lambda_2}\, |\mathscr{F}(\mathfrak{r},\mu^*(\mathfrak{r}))| \int_0^1 \int_0^{\tau} \frac{\tau}{\tau+\varrho} \left[\left|\phi\left(\varrho, \nu(\varrho), \int_0^{\varrho} \frac{\varrho}{\varrho+\varsigma}\psi(\varsigma,\mu(\varsigma))d\varsigma\right)\right.\right. \\
&- \phi\left(\varrho, \nu(\varrho), \int_0^{\varrho} \frac{\varrho}{\varrho+\varsigma}\psi(\varsigma,\mu^*(\varsigma))d\varsigma\right)\Bigg| \\
&+ \left|\phi\left(\varrho, \nu(\varrho), \int_0^{\varrho} \frac{\varrho}{\varrho+\varsigma}\psi(\varsigma,\mu^*(\varsigma))d\varsigma\right) - \phi\left(\varrho, \nu^*(\varrho), \int_0^{\varrho} \frac{\varrho}{\varrho+\varsigma}\psi(\varsigma,\mu^*(\varsigma))d\varsigma\right)\right|\Bigg] d\varrho d\tau \\
&+ |F(\mathfrak{r},\mu(\mathfrak{r})) - F(\mathfrak{r},\mu^*(\mathfrak{r}))| \int_0^{\eta} \frac{(\eta-\tau)^{\gamma-1}}{\Gamma(\gamma)} \int_0^{\tau} \frac{\tau}{\tau+\varrho} \left|\phi\left(\varrho, \nu(\varrho), \int_0^{\varrho} \frac{\varrho}{\varrho+\varsigma}\, \psi(\varsigma,\mu(\varsigma))\, d\varsigma\right)\right| d\varrho d\tau \\
&+ |\mathscr{F}(\mathfrak{r},\mu^*(\mathfrak{r}))| \int_0^{\eta} \frac{(\eta-\tau)^{\gamma-1}}{\Gamma(\gamma)} \int_0^{\tau} \frac{\tau}{\tau+\varrho} \left[|\phi\left(\varrho, \nu(\varrho), \int_0^{\varrho} \frac{\varrho}{\varrho+\varsigma}\psi(\varsigma,\mu(\varsigma))d\varsigma\right)\right. \\
&- \phi\left(\varrho, \nu(\varrho), \int_0^{\varrho} \frac{\varrho}{\varrho+\varsigma}\psi(\varsigma,\mu^*(\varsigma))d\varsigma\right)\Bigg| \\
&+ \left|\phi\left(\varrho, \nu(\varrho), \int_0^{\varrho} \frac{\varrho}{\varrho+\varsigma}\psi(\varsigma,\mu^*(\varsigma))d\varsigma\right) - \phi(\varrho, \nu^*(\varrho), \int_0^{\varrho} \frac{\varrho}{\varrho+\varsigma}\psi(\varsigma,\mu^*(\varsigma))d\varsigma\right)\Bigg|\Bigg] d\varrho d\tau \\
&+ \frac{1}{\lambda_2}[|\mathscr{F}(\mathfrak{r},\mu^*(\mathfrak{r}))\, \theta(\mathfrak{r},\mu^*(\mathfrak{r})) - \mathscr{F}(\mathfrak{r},\mu(\mathfrak{r}))\, \theta(\mathfrak{r},\mu(\mathfrak{r}))|]
\end{aligned}$$

$$
\begin{aligned}
&\leq |\mathscr{F}(\mathfrak{r},\mu(\mathfrak{r})) - \mathscr{F}(\mathfrak{r},\mu^*(\mathfrak{r}))| \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\tau)^{\gamma-1}}{\Gamma(\gamma)} \int_0^{\tau} \frac{\tau}{\tau+\varrho} \big[m(\varrho) + k_1(\varrho)\big(|\varpi(\varrho,\nu(\varrho),\mu(\varrho))| \\
&+ \int_0^{\varrho} \frac{\varrho}{\varrho+\varsigma} \left|\psi(\varsigma,\mu(\varsigma))\right| d\varsigma\big)\big]\, d\varrho\, d\tau \\
&+ |\mathscr{F}(\mathfrak{r},\mu^*(\mathfrak{r}))| \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\tau)^{\gamma-1}}{\Gamma(\gamma)} \int_0^{\tau} \frac{\tau}{\tau+\varrho} \big(k_1(\varrho)\big(|\nu(\varrho)-\nu^*(\varrho)| \\
&+ \int_0^{\varrho} \frac{\varrho}{\varrho+\varsigma} \left|\psi(\varsigma,\mu(\varsigma)) - \psi(\varsigma,\mu^*(\varsigma))\right| d\varsigma\big) d\varrho\big) d\tau + \frac{\lambda_1}{\lambda_2} |\mathscr{F}(\mathfrak{r},\mu(\mathfrak{r})) - \mathscr{F}(\mathfrak{r},\mu^*(\mathfrak{r}))| \\
&\times \int_0^{\varrho} \frac{(\varrho-\tau)^{1-\gamma}}{\Gamma(2-\gamma)} \int_0^{\tau} \frac{\tau}{\tau+\varrho} \big[m(\varrho) + k_1(\varrho)\big(|\varpi(\varrho,\nu(\varrho),\mu(\varrho))| \\
&+ \int_0^{\varrho} \frac{\varrho}{\varrho+\varsigma} k_1(\varsigma) \left|\psi(\varsigma,\mu(\varsigma))\right| d\varsigma\big)\big]\, d\varrho\, d\tau \\
&+ \frac{\lambda_1}{\lambda_2} |\mathscr{F}(\mathfrak{r},\mu^*(\mathfrak{r}))| \int_0^1 \int_0^{\tau} \frac{\tau}{\tau+\varrho} \big(k_1(\varrho)\big(|\nu(\varrho)-\nu(\varrho)^*| \\
&+ \int_0^{\varrho} \frac{\varrho}{\varrho+\varsigma} \left|\psi(\varsigma,\mu_1(\varsigma)) - \psi(\varsigma,\mu^*(\varsigma))\right| d\varsigma\big) d\varrho\big) ds \\
&+ |\mathscr{F}(\mathfrak{r},\mu(\mathfrak{r})) - \mathscr{F}(\mathfrak{r},\mu^*(\mathfrak{r}))| \int_0^{\eta} \frac{(\eta-\tau)^{\gamma-1}}{\Gamma(\gamma)} \int_0^{\tau} \frac{\tau}{\tau+\varrho} \big[m(\varrho) + k_1(\varrho)\big(|\varpi(\varrho,\nu(\varrho),\mu(\varrho))| \\
&+ \int_0^{\varrho} \frac{\varrho}{\varrho+\varsigma} \left|\psi(\varsigma,\mu(\varsigma)))\right| d\varsigma\big)\big]\, d\varrho\, d\tau \\
&+ |\mathscr{F}(\mathfrak{r},\mu^*(\mathfrak{r}))| \int_0^{\eta} \frac{(\eta-\tau)^{\gamma-1}}{\Gamma(\gamma)} \int_0^{s} \frac{s}{s+\varrho} \big(k_1(\varrho)\big(|\nu(\varrho)-\nu(\varrho)^*| \\
&+ \int_0^{\varrho} \frac{\varrho}{\varrho+\varsigma} \left|\psi(\varsigma,\mu(\varsigma)) - \psi(\varsigma,\mu^*(\varsigma))\right| d\varsigma\big) d\varrho\big) d\tau \\
&+ \frac{1}{\lambda_2} \big[|\mathscr{F}(\mathfrak{r},\mu^*(\mathfrak{r}))|\, |\theta(\mathfrak{r},\mu_2(\mathfrak{r})) - \theta(\mathfrak{r},\mu_1(\mathfrak{r}))| + |\mathscr{F}(\mathfrak{r},\mu^*(\mathfrak{r})) - \mathscr{F}(\mathfrak{r},\mu(\mathfrak{r}))|\, |\theta(\mathfrak{r},\mu(\mathfrak{r}))|\big] \\
&\leq \|\omega\| |\mu(\mathfrak{r}) - \mu^*(\mathfrak{r})| \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\tau)^{\gamma-1}}{\Gamma(\gamma)} \int_0^1 \frac{\tau}{\tau+\varrho} \big[|m(\varrho)| \\
&+ |k_1(\varrho)|\big(|\delta(\varrho)| + \int_0^1 \frac{\varrho}{\varrho+\varsigma} |k_2(\varsigma)||\mu(\varsigma)|\, d\varsigma\big)\big]\, d\varrho\, d\tau \\
&+ \big[\|\omega\|\, |\mu^*(\mathfrak{r})| + G\big] \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\tau)^{\gamma-1}}{\Gamma(\gamma)} \int_0^{\tau} \frac{\tau}{\tau+\varrho} \big(k_1(\varrho)\big(\gamma + \int_0^{\varrho} \frac{\varrho}{\varrho+\varsigma} k_2(\varsigma)\, |\mu(\varsigma) - \mu^*(\varsigma)| d\varsigma\big) d\varrho\big) d\tau \\
&+ \frac{\lambda_1}{\lambda_2} \|\omega\|\, |\mu(\mathfrak{r}) - \mu^*(\mathfrak{r})| \\
&\times \int_0^1 \int_0^1 \frac{\tau}{\tau+\varrho} \big[|m(\varrho)| + |k_1(\varrho)|\big(|\delta(\varrho)| + \int_0^1 \frac{\varrho}{\varrho+\varsigma}\ |k_2(\varsigma)||\mu(\varsigma)|\, d\varsigma\big)\big]\, d\varrho\, d\tau \\
&+ \frac{\lambda_1}{\lambda_2} \big[\|\omega\|\, |\mu^*(\mathfrak{r})| + G\big] \int_0^1 \int_0^{\tau} \frac{\tau}{\tau+\varrho} \big(|k_1(\varrho)|\big(\gamma + \int_0^{\varrho} \frac{\varrho}{\varrho+\varsigma} |k_2(\varsigma)|\, |\mu(\varsigma) - \mu^*(\varsigma)| d\varsigma\big) d\varrho\big) d\tau \\
&+ \|\omega\| |\mu(\mathfrak{r}) - \mu^*(\mathfrak{r})| \\
&\times \int_0^{\eta} \frac{(\eta-\tau)^{\gamma-1}}{\Gamma(\gamma)} \int_0^1 \frac{\tau}{\tau+\varrho} \big[|m(\varrho)| + |k_1(\varrho)|\big(|\delta(\varrho)| + \int_0^1 \frac{\varrho}{\varrho+\varsigma}\ |k_2(\varsigma)||\mu(\varsigma)|\, d\varsigma\big)\big]\, d\varrho\, d\tau \\
&+ \big[\|\omega\||\mu^*(\mathfrak{r})| + G\big] \int_0^{\eta} \frac{(\eta-\tau)^{\gamma-1}}{\Gamma(\gamma)} \int_0^1 \frac{\tau}{\tau+\varrho} \big(k_1(\varrho)\big(\gamma + \int_0^{\varrho} \frac{\varrho}{\varrho+\varsigma} k_2(\varsigma)\, |\mu(\varsigma) - \mu^*(\varsigma)| d\varsigma\big) d\varrho\big) d\tau \\
&+ \frac{k_3\, |\mu^*(\mathfrak{r}) - \mu(\mathfrak{r})|}{\lambda_2}\ \big[\|\omega\||\mu^*(\mathfrak{r})| + G\big] + \|\omega\||\mu(\mathfrak{r}) - \mu^*(\mathfrak{r})|\ \big[k_3|\mu(\mathfrak{r})| + \Theta\big]
\end{aligned}
$$

$$\leq \frac{\|\omega\| \, \|\mu-\mu^*\|}{\Gamma(\gamma+1)} [m + k_1 (\|\delta\| + k_2 \|\mu\|)] + \frac{[\|\omega\| \, \|\mu^*\| + G]}{\Gamma(\gamma+1)} (k_1 \, (\Upsilon + k_2 \, \|\mu-\mu^*\|)$$
$$+ \frac{\lambda_1}{\lambda_2} \|\omega\| \, \|\mu-\mu^*\| [m + k_1 (\|\delta\| + k_2 \, \|\mu\|)]$$
$$+ \frac{\lambda_1}{\lambda_2} [\|\omega\| \, |\mu^*(t)| + G] \, (k_1 \, (\Upsilon + k_2 \|\mu-\mu^*\|))$$
$$+ \frac{\eta^\gamma \, \|\omega\| \, \|\mu-\mu^*\|}{\Gamma(\gamma+1)} [m + k_1 \, (\|\delta\| + k_2 \, \|\mu\|)] + \frac{\eta^\gamma \, [\|\omega\| \, \|\mu^*\| + G]}{\Gamma(\gamma+1)} (k_1 \, (\Upsilon + k_2 \, |\mu-\mu^*\|))$$
$$+ \frac{1}{\lambda_1} [\, k_3 \, \|\mu-\mu^*\| \, [\|\omega\| \|\mu^*\| + G] + \|\omega\| \, \|\mu-\mu^*\| \, [k_3 \|\mu\| + \Theta]].$$

Taking the supremum over $\mathfrak{r} \in I$, we have

$$\begin{aligned}
\|\mu-\mu^*\| \leq \; & [\frac{1}{\Gamma(\gamma+1)} + \frac{\lambda_1}{\lambda_2} + \frac{\eta^\gamma}{\Gamma(\gamma+1)}] \, \|\omega\| \, \|\mu-\mu^*\| [m + k_1 \, (\|\delta\| + k_2 \, r)] \\
& + [\frac{1}{\Gamma(\gamma+1)} + \frac{\lambda_1}{\lambda_2 \Gamma(3-\gamma)} + \frac{\eta^\gamma}{\Gamma(\gamma+1)}] \, [\|\omega\| \, r + G] (k_1 \, (\Upsilon + k_2 \|\mu-\mu^*\|)) \\
& + \|\mu-\mu^*\| \, [\, k_3 \, [\|\omega\| \|\mu^*\| + G] + \|\omega\| \, [k_3 \|\mu\| + \Theta]] \\
& \leq \; \|\mu-\mu^*\| \Big( \Lambda \, [\|\omega\| (m + k_1 \, (\|\delta\| + k_2 \, r)) + (\|\omega\| \, r + G) k_1 \, k_2] \\
& + \frac{[\, k_3 \, [\|\omega\| \|\mu^*\| + G] + \|\omega\| \, [k_3 \|\mu_1\| + \Theta]]}{\lambda_2} \Big) + [\|\omega\| \, r + G] \Upsilon \, \Lambda \, k_1
\end{aligned}$$

and

$$\|\mu-\mu^*\| \leq \frac{\|\omega\| r + G] \Upsilon \Lambda k_1}{1 - \left( \Lambda [\|\omega\| (m + k_1 (\|\delta\| + k_2 r)) + (\|\omega\| r + G) k_1 k_2] + \frac{k_3 [\|\omega\| \|\mu^*\| + G] + \|\omega\| \, [k_3 \|\mu_1\| + \Theta]}{\lambda_2} \right)}$$
$$= \epsilon.$$

The previous inequality leads to the following result:

$$\|\mu-\mu^*\| \leq \epsilon.$$

This demonstrates the solution's continuous dependence on the control variable function $\nu$. □

## 3. Set-Valued Problem

The study of inclusion problems has drawn much interest based on their extensive applications and actual problems [15,20,21]. Regarding the differential inclusion problems and some results of existence, see [22–25].

$(\mathscr{H}_1^{**})$ Let $\Phi : I \times \mathbb{R} \times \mathbb{R} \to 2^{\mathbb{R}}$ be non-empty and convex and let subset $\forall \, (\mathfrak{r}, \nu, \mu) \in I \times \mathbb{R} \times \mathbb{R}$, where

- $(i)$ $\Phi(\mathfrak{r}, ., .)$ is upper semicontinuous in $(\nu, \mu) \in \mathbb{R} \times \mathbb{R}$, $\forall \mathfrak{r} \in I$.
- $(ii)$ $\Phi(., \nu, \mu)$ is measurable in $\mathfrak{r} \in I$, $\forall (\nu, \mu) \in \mathbb{R} \times \mathbb{R}$.
- $(iii)$ There exist $m, k_1 : I \to I$, where $m, k_1 \in L^1(I)$ with

$$|\Phi(\mathfrak{r}, \nu, \tau)| = sup\{|\phi| : \phi \in \Phi(\mathfrak{r}, \nu, \tau)\} \leq m(\mathfrak{r}) + k_1(\mathfrak{r}) (|\nu| + |\tau|), \;\; \mathfrak{r} \in I$$

and

$$\int_0^1 \frac{1}{\tau + \varrho} \, |m(\varrho)| \, d\varrho \leq m, \;\; \text{and} \;\; \int_0^1 \frac{1}{\tau + \varrho} \, |k_1(\varrho)| \, d\varrho \leq k.$$

$(\mathcal{H}_3^*)$ Let $\Theta : I \times \mathbb{R} \to 2^{\mathbb{R}}$ be a Lipschitzian set-valued function with a nonempty compact convex subset of $2^{\mathbb{R}}$, where

$$\|\Theta(\mathfrak{r}, \mu_1) - \Theta(\mathfrak{r}, \mu_2)\| \leq k_3(\mathfrak{r})|\mu_1 - \mu_2|.$$

**Remark 2.** *Obviously, we can deduce that, as shown in Remark 1 [9], there exists a Carathéodory mapping $\phi \in \Phi$ [26] that is measurable in $\mathfrak{r} \in I$, $\forall\, \nu, \mu \in \mathbb{R}$ and continuous in $\nu, \mu \in \mathbb{R}, \forall\, \mathfrak{r} \in I$,*

$$|\phi(\mathfrak{r}, \nu.\mu)| \leq m(\mathfrak{r}) \ + \ k_1(\mathfrak{r})(|\nu| + |\mu|), \ \ \mathfrak{r} \in I.$$

*In addition, the set of Lipschitzian selections $S_\Theta$ is nonempty [26] and $\theta \in S_\Theta$ meets*

$$|\theta(\mathfrak{r}, \mu_1) - \theta(\mathfrak{r}, \mu_2)| \leq k_3(\mathfrak{r})|\mu_1 - \mu_2|,$$

*Then,*

$$|\theta(\mathfrak{r}, \mu)| \leq k_1(\mathfrak{r})|\mu| + \Theta, \quad \Theta = \sup_{\mathfrak{r} \in I} |\theta(\mathfrak{r}, 0)|$$

*and satisfies the nonlocal problem (6) and (7) with feedback control (8).*

*Let $\Omega(\mathfrak{r}, \nu(\mathfrak{r}), \mu(\mathfrak{r})) : I \times \mathbb{R}^+ \times \mathbb{R}^+ \to 2^{\mathbb{R}^+}$ meet the mentioned conditions:*

*(a)* *The set $\Omega(\mathfrak{r}, \nu(\mathfrak{r}), \mu(\mathfrak{r}))$ is a non-empty, closed and convex subset for all $(\mathfrak{r}, \nu(\mathfrak{r}), \mu(\mathfrak{r})) \in I \times \mathbb{R}^+ \times \mathbb{R}^+$.*
*(b)* *$\Omega(\mathfrak{r}, \nu(\mathfrak{r}), \mu(\mathfrak{r}))$ is upper semicontinuous in $\nu, \mu \in \mathbb{R}^+$ for each $\mathfrak{r} \in I$.*
*(c)* *$\Omega(., \nu(\mathfrak{r}), \mu(\mathfrak{r}))$ is measurable in $\mathfrak{r} \in I$ for each $\nu, \mu \in \mathbb{R}^+$.*
*(d)* *There exists two measurable and bounded functions $\delta : I \to \mathbb{R}$, with norm $\|\delta\|$, where*

$$|\Omega(\mathfrak{r}, \nu(\mathfrak{r}), \mu(\mathfrak{r}))| = sup\{|\eta| : \eta \in \Omega(\mathfrak{r}, \nu(\mathfrak{r}), \mu(\mathfrak{r}))\} \leq \delta(\mathfrak{r}), \ \ \mathfrak{r} \in I,$$

*with $\delta = \max_{\mathfrak{r} \in I}\{\|\delta\|\}$.*

**Remark 3.** *We can infer from assumption (i) that the set of selections $S_\Omega$ $(i = 1, 2, ..., k)$ of the set-valued function $\Omega$ is nonempty and there exists a Carathéodory function $\varpi \in \Omega$ (see [26]) such that*

$$|\varpi(\mathfrak{r}, \nu(\mathfrak{r}), \mu(\mathfrak{r}))| \leq \delta(\mathfrak{r}),$$

*fulfilling the implicit equation*

$$\nu(\tau) \ = \ \varpi(\mathfrak{r}, \nu(\mathfrak{r}), \mu(\mathfrak{r})), \quad \mathfrak{r} \in I. \tag{17}$$

*Thus, any solution of problem (6) and (7) with multi-valued feedback control (8) is a solution of problem (6) and (7) with feedback control (17).*

### 3.1. Existence Results

Now, based on the main findings in Section 2, we present in the following the results obtained for the nonlocal hybrid modeling of a heat controller (3) via the multi-valued condition (4) with feedback control (5).

**Theorem 4.** *Let the assumptions $(\mathcal{H}_1^{**})$,$(\mathcal{H}_3^*)$ and $(\mathcal{H}_2^*)$,$(\mathcal{H}_4)$, $(\mathcal{H}_5)$ hold. Then, the problem (3) and (4) has one solution, $\mu \in C(I, \mathbb{R})$.*

In the aim of demonstrating uniqueness result of (3)–(5), we replace the assumption $(\mathcal{H}_1)$ by

$(\mathscr{H}_1^{***})$. Let $\Phi : I \times \mathbb{R} \times \mathbb{R} \to 2^{\mathbb{R}}$ be a Lipschitzian multi-valued mapping with a nonempty convex compact subset of $2^R$, with

$$\|\Phi(\mathfrak{r}, \nu_1, \mu_1) - \Phi(\mathfrak{r}, \nu_2, \mu_2)\| \le k_1(\mathfrak{r})(|\nu_1 - \nu_2| + |\mu_1 - \mu_2|).$$

From these assumptions, we can observe that $(\mathscr{H}_1^{**})$ is held. In addition, the Lipschitzian selection set $S_\Phi$ is nonempty ([26]) and $\phi \in S_\Phi$ meets

$$|\phi(\mathfrak{r},, \nu_1, \mu_1) - \phi(\mathfrak{r},, \nu_2, \mu_2)| \le k_1(\mathfrak{r})(|\nu_1 - \nu_2| + |\mu_1 - \mu_2|),$$

then, we have

$$|\phi(\mathfrak{r}, \nu, \mu)| \le k_1(\mathfrak{r})(|\nu| + |\mu|) + m, \quad m = \sup_{\mathfrak{r} \in I} |\phi(\mathfrak{r}, 0, 0)|$$

**Theorem 5.** *Let Theorem 4 be verified and replace assumption $(\mathscr{H}_1^{**})$ with assumption $(\mathscr{H}_1^{***})$. Then, inclusion problem (3)–(5) has a unique solution, $x \in C(I, \mathbb{R})$.*

*3.2. Continuous Dependency on the Sets of Selections*

**Definition 2.** *The solution of problem (7) and (6), with multi-valued feedback control (17) depends continuously on the set $S_\Phi$. If $\forall\ \epsilon > 0,\ \ \exists\ \Upsilon > 0$, such that*

$$|\phi(\mathfrak{r}, \nu, \mu) - \phi^*(\mathfrak{r}, \nu, \mu)| \quad < \quad \Upsilon,\ \ \phi,\ \phi^* \in S_\Phi\ ,\ \ \mathfrak{r} \in [0, 1],$$

*then, $\|\mu - \mu^*\| < \epsilon$.*

**Theorem 6.** *Let Theorem 5 be verified. Then, the solution of (13) depends continuously on the set $S_\Phi$ of all Lipschitzian selections of $\phi$.*

**Proof.** Let the functions $\mu(\mathfrak{r})$ and $\mu^*(\mathfrak{r})$ of (13), correspond to $\phi,\ \phi^*\ \in S_\Phi$, respectively; then,

$$\begin{aligned}
&|\mu(\mathfrak{r}) - \mu^*(\mathfrak{r})| \\
&\le |\mathscr{F}(\mathfrak{r}, \mu(\mathfrak{r})) - \mathscr{F}(\mathfrak{r}, \mu^*(\mathfrak{r}))| \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\tau)^{\gamma-1}}{\Gamma(\gamma)} \int_0^{\tau} \frac{\tau}{\tau+\varrho} \left| \phi\left(\varrho, \nu(\varrho), \int_0^{\varrho} \frac{\varrho}{\varrho+\varsigma}\ \psi(\varsigma, \mu(\varsigma))\ d\varrho\right) \right| d\varrho d\tau \\
&+ |\mathscr{F}(\mathfrak{r}, \mu^*(\mathfrak{r}))| \int_0^{\mathfrak{r}} \frac{(\mathfrak{r}-\tau)^{\gamma-1}}{\Gamma(\gamma)} \int_0^{\tau} \frac{\tau}{\tau+\varrho} \Bigg[ \phi\left(\varrho, \nu(\varrho), \int_0^{\varrho} \frac{\varrho}{\varrho+\varsigma} \psi(\varsigma, \mu(\varsigma)) d\varsigma\right) \\
&- \phi^*\left(\varrho, \nu(\varrho), \int_0^{\varrho} \frac{\varrho}{\varrho+\varsigma} \psi(\varsigma, \mu^*(\varsigma)) d\varsigma\right) \Bigg] d\varrho d\tau \\
&+ \frac{\lambda_1}{\lambda_2}\ |\mathscr{F}(\mathfrak{r}, \mu(\mathfrak{r})) - \mathscr{F}(\mathfrak{r}, \mu^*(\mathfrak{r}))| \int_0^1 \int_0^{\tau} \frac{\tau}{\tau+\varrho} \left| \phi\left(\varrho, \nu(\varrho), \int_0^{\varrho} \frac{\varrho}{\varrho+\varsigma}\ \psi(\varsigma, \mu(\varsigma))\ d\varsigma\right) \right| d\varrho d\tau \\
&+ \frac{\lambda_1}{\lambda_2}\ |\mathscr{F}(\mathfrak{r}, \mu^*(\mathfrak{r}))| \int_0^1 \int_0^{\tau} \frac{\tau}{\tau+\varrho} \Bigg[ \phi\left(\varrho, \nu(\varrho), \int_0^{\tau} \frac{\varrho}{\varrho+\varsigma} \psi(\varsigma, \mu(\varsigma)) d\varsigma\right) \\
&- \phi^*\left(\varrho, \nu(\varrho), \int_0^{\varrho} \frac{\varrho}{\varrho+\varsigma} \psi(\varsigma, \mu^*(\varsigma)) d\varsigma\right) \Bigg] d\varrho d\tau \\
&+ |\mathscr{F}(\mathfrak{r}, \mu(\mathfrak{r})) - \mathscr{F}(\mathfrak{r}, \mu^*(\mathfrak{r}))| \int_0^{\eta} \frac{(\eta-\tau)^{\gamma-1}}{\Gamma(\gamma)} \int_0^{\tau} \frac{\tau}{\tau+\varrho} \left| \phi\left(\varrho, \nu(\varrho), \int_0^{\varrho} \frac{\varrho}{\varrho+\varsigma}\ \psi(\varsigma, \mu(\varsigma))\ d\varsigma\right) \right| d\varrho d\tau \\
&+ |\mathscr{F}(\mathfrak{r}, \mu^*(\mathfrak{r}))| \int_0^{\eta} \frac{(\eta-\tau)^{\gamma-1}}{\Gamma(\gamma)} \int_0^{\tau} \frac{\tau}{\tau+\varrho} \Bigg[ \phi\left(\varrho, \nu(\varrho), \int_0^{\varrho} \frac{\varrho}{\varrho+\varsigma} \psi(\varsigma, \mu(\varsigma)) d\varsigma\right) \\
&- \phi^*\left(\varrho, \nu(\varrho), \int_0^{\varrho} \frac{\varrho}{\varrho+\varsigma} \psi(\varsigma, \mu^*(\varsigma)) d\varsigma\right) \Bigg] d\varrho d\tau
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\lambda_2}|\mathcal{F}(\mathfrak{r},\mu^*(\mathfrak{r}))\,\theta(\mathfrak{r},\mu^*(\mathfrak{r}))-\mathcal{F}(\mathfrak{r},\mu(\mathfrak{r}))\,\theta(\mathfrak{r},\mu(\mathfrak{r}))| \\
&\leq \|\omega\|\,|\mu(\mathfrak{r}))-\mu^*(\mathfrak{r})|\int_0^{\mathfrak{r}}\frac{(\mathfrak{r}-\tau)^{\gamma-1}}{\Gamma(\gamma)}\int_0^1\left[|m(\varrho)|+|k_1(\varrho)|\left(\|\delta\|+\ k_2\|\mu\|\right)\right]d\varrho\, d\tau \\
&+\left[\|\omega\|\,|\mu^*(\mathfrak{r})|+G\right]\int_0^{\mathfrak{r}}\frac{(\mathfrak{r}-\tau)^{\gamma-1}}{\Gamma(\gamma)}\int_0^{\tau}\left(\gamma+|k_1(\varrho)|\,\left(\|\delta\|+\ k_2\,\|\mu-\mu^*\|\right)d\varrho\right)d\tau \\
&+\frac{\lambda_1}{\lambda_2}\ \|\omega\|\,|\mu(\mathfrak{r}))-\mu^*(\mathfrak{r})|\int_0^1\int_0^{\mathfrak{r}}\left[|m(\varrho)|+|k_1(\varrho)|\left(\|\delta\|+\ k_2\,\|\mu_1\|\right)\right]d\varrho\, d\tau \\
&+\frac{\lambda_1}{\lambda_2}\left[\|\omega\|\,|\mu^*(\mathfrak{r})|+G\right]\int_0^1\int_0^{\tau}\left(\Upsilon+|k_1(\varrho)|\left(\|\delta\|+\ k_2\|\mu-\mu^*\|\right)d\varrho\right)d\tau \\
&+\|\omega\|\,|\mu(\mathfrak{r}))-\mu^*(\mathfrak{r})|\int_0^{\eta}\frac{(\eta-\tau)^{\gamma-1}}{\Gamma(\gamma)}\int_0^1\left[|m(\varrho)|+|k_1(\varrho)|\,\left(\|\delta\|+\ k_2\,\|\mu\|\right)\right]d\varrho\, d\tau \\
&+\ \left[\|\omega\|\,|\mu^*(\mathfrak{r})|+G\right]\int_0^{\eta}\frac{(\eta-\tau)^{\gamma-1}}{\Gamma(\gamma)}\int_0^1\left(\Upsilon+|k_1(\varrho)|\,\left(\|\delta\|+\ k_2\,\|\mu-\mu^*\|\right)d\varrho\right)d\tau \\
&+\frac{1}{\lambda_2}\left[\,k_3\ \ \|\mu-\mu^*\|\,\left[\|\omega\|\|\mu^*\|+G\right]+\ \|\omega\|\,\|\mu-\mu^*\|\,\left[k_3\|\mu\|+\Theta\right]\right] \\
&\leq\frac{\|\omega\|\,\|\mu-\mu^*\|}{\Gamma(\gamma+1)}\left[m+k_1\left(\|\delta\|+\ k_2\|\mu\|\right)\right]+\frac{\left[\|\omega\|\,\|\mu^*\|+G\right]}{\Gamma(\gamma+1)}\left(\Upsilon+k_1\,\left(\|\delta\|+\ k_2\,\|\mu-\mu^*\|\right)\right. \\
&+\frac{\lambda_1}{\lambda_2}\,\|\omega\|\,\|\mu-\mu^*\|\left[m+k_1\left(\|\delta\|+\ k_2\,\|\mu\|\right)\right] \\
&+\frac{\lambda_1}{\lambda_2}\left[\|\omega\|\,|\mu^*(t)|+G\right]\,\left(\Upsilon+k_1\,\left(\|\delta\|+\ k_2\|\mu-\mu^*\|\right)\right) \\
&+\frac{\eta^{\gamma}\,\|\omega\|\,\|\mu-\mu^*\|}{\Gamma(\gamma+1)}\left[m+k_1\,\left(\|\delta\|+\ k_2\,\|\mu\|\right)\right]+\frac{\eta^{\gamma}\,\left[\|\omega\|\,\|\mu^*\|+G\right]}{\Gamma(\gamma+1)}\left(\Upsilon+k_1\,\left(\|\delta\|+\ k_2\,|\mu-\mu^*\|\right)\right) \\
&+\frac{1}{\lambda_2}\left[\,k_3\ \ \|\mu-\mu^*\|\,\left[\|\omega\|\|\mu^*\|+G\right]+\ \|\omega\|\,\|\mu-\mu^*\|\,\left[k_3\|\mu\|+\Theta\right]\right].
\end{aligned}
$$

For $\mathfrak{r}\in[0,1]$, we obtain

$$
\begin{aligned}
\|\mu-\mu^*\|\leq\ &\left[\frac{1}{\Gamma(\gamma+1)}+\frac{\lambda_1}{\lambda_2}+\frac{\eta^{\gamma}}{\Gamma(\gamma+1)}\right]\|\omega\|\,\|\mu-\mu^*\|\left[m+k_1\,\left(\|\delta\|+\ k_2\,r\right)\right] \\
&+\ \left[\frac{1}{\Gamma(\gamma+1)}+\frac{\lambda_1}{\lambda_2\Gamma(3-\gamma)}+\frac{\eta^{\gamma}}{\Gamma(\gamma+1)}\right]\ \left[\|\omega\|\,r+G\right]\left(\Upsilon+k_1\,\left(\|\delta\|+\ k_2\|\mu-\mu^*\|\right)\right) \\
&+\frac{\|\mu-\mu^*\|}{\lambda_2}\,\left[\,k_3\ \left[\|\omega\|\|\mu^*\|+G\right]+\ \|\omega\|\ \left[k_3\|\mu\|+\Theta\right]\right] \\
&\leq\ \|\mu-\mu^*\|\Bigg(\Lambda\left[\omega(m+k_1\,\left(\|\delta\|+\ k_2\,r\right))+(\|\omega\|\,r+G)\left(\Upsilon+k_1\,\left(\|\delta\|+\ k_2\right)\right)\right] \\
&+\frac{\left[\,k_3\ \left[\|\omega\|\|\mu_2\|+G\right]+\ \|\omega\|\ \left[k_3\|\mu_1\|+\Theta\right]\right]}{\lambda_2}\Bigg)+\|\omega\|\,r+G]\left[\Upsilon+k_1\|\delta\|\right]\Lambda
\end{aligned}
$$

and

$$
\begin{aligned}
\|\mu-\mu^*\|\leq&\frac{\left[\|\omega\|\,r+G\right]\left[\Upsilon+k_1\|\delta\|\right]\Lambda}{1-\left(\Lambda[\|\omega\|(m+k_1(\|\delta\|+k_2r))+(\|\omega\|r+G)k_1k_2]+\frac{[k_3[\|\omega\|r+G]+\|\omega\|[k_3r+\Theta]]}{\lambda_2}\right)} \\
&=\epsilon
\end{aligned}
$$

The previous inequality leads to the following result:

$$\|\mu-\mu^*\|\leq\epsilon.$$

This demonstrates the solution of the problem (7) and (6) with multi-valued feedback control (17) is continuously dependent on the set $S_{\Phi}$. $\square$

*3.3. Example*

In accordance with the indicated hybrid BVP (3) and (4), we take into account the fractional order hybrid inclusion problem

$$-{}^{c}\mathscr{D}^{\gamma}\left(\frac{\mu(\mathfrak{r})}{\frac{e^{-ln^2(\mathfrak{r}+1)}|\mu(\mathfrak{r})|}{1+|\mu(\mathfrak{r})|}+8}\right) \in \left[\int_0^{\mathfrak{r}} \frac{\tau}{\mathfrak{r}+\tau}\left[\tau + \frac{\int_0^{\tau}\nu(\tau)\,\sin\frac{|\mu(\varrho)|}{1+|\mu(\varrho)|}d\varrho}{2(1+\int_0^{\tau}\nu(\tau)\,\sin\frac{|\mu(\varrho)|}{1+|\mu(\varrho)|}d\varrho)}\right]d\tau,\,0\right],\quad \mathfrak{r}\in I \tag{18}$$

via the nonlocal conditions

$$\begin{cases} \mathscr{D}\left(\frac{\mu(\mathfrak{r})}{\frac{e^{-ln^2(\mathfrak{r}+1)}|\mu|}{100(1+|\mu|)}+8}\right)\Bigg|_{\mathfrak{r}=0} = 0, \\ \lambda_1\,{}^{c}\mathscr{D}^{\gamma-1}\left(\frac{\mu(\mathfrak{r})}{\frac{e^{-ln^2(\mathfrak{r}+1)}|\mu|}{100(1+|\mu|)}+8}\right)\Bigg|_{\mathfrak{r}=1} + \lambda_2\left(\frac{\mu(\mathfrak{r})}{\frac{e^{-ln^2(\mathfrak{r}+1)}|\mu|}{100(1+|\mu|)}+8}\right)\Bigg|_{\mathfrak{r}=\eta} \in \left[\frac{\mathfrak{r}}{2}+\frac{\mu(\mathfrak{r})}{4+\mathfrak{r}},0\right], \end{cases} \tag{19}$$

with multi-valued feedback

$$\mu(\mathfrak{r}) \in [0.1\,\mu(\mathfrak{r}) + \frac{1}{200}\cos(\mathfrak{r}) + e^{\frac{-3}{2}\mathfrak{r}}\,\nu(\mathfrak{r}),0]. \tag{20}$$

Let $\gamma = \frac{7}{4}, \gamma - 1 = \frac{3}{4}$, $\eta = 0.89$, and $\lambda_1 = \lambda_2 = \frac{9}{5}$.
Define $g(\mathfrak{r},\mu(\mathfrak{r})) = \frac{e^{-ln^2(\mathfrak{r}+1)}|u|}{100(1+|u|)} + 8$ and the multi-valued map $\Phi$ by

$$\Phi(\mathfrak{r},\nu,\mu) = \left[\mathfrak{r} + \frac{\int_0^{\mathfrak{r}}\nu(\mathfrak{r})\,\sin\frac{|\mu(\mathfrak{r})|}{1+|\mu(\mathfrak{r})|}d\tau}{2(1+\int_0^{\mathfrak{r}}\nu(\mathfrak{r})\,\sin\frac{|\mu(\tau)|}{1+|\mu(\tau)|}d\tau)}\,,0\right] \text{ and } \theta(\mathfrak{r},x(\mathfrak{r})) = \frac{\mathfrak{r}}{2}+\frac{\mu(\mathfrak{r})}{4+\mathfrak{r}}.$$

If $w(\mathfrak{r}) = \frac{e^{-ln^2(\mathfrak{r}+1)}}{100}$, then $\|w\| = \frac{1}{100}$, $m = 0.5\,k = \frac{1}{2}, k^* = \frac{1}{2},\ k_3 = \frac{1}{4}, G = \frac{1}{2}$, and $\theta = \frac{1}{2}$. By using the above relations, we get $\Lambda \simeq 1.9468834$. Hence, the above data satisfy the condition of Theorem 2:

$$\begin{aligned}\Lambda\,\big[\|\omega\|(m+k_1\,(\|\delta\|+\,k_2\,r))\quad &+\quad (\|\omega\|\,r+G)k_1\,\,k_2\big] + \frac{1}{\lambda_2}\big[\,k_3\,\,\big[\|\omega\|r+G\big] \\ &+\quad \|\omega\|\,\,\big[k_3 r+\Theta\big]\big] \simeq 0.5826955222 < 1.\end{aligned}$$

Using Theorem 5, then the problem (18) and (19) with multi-valued feedback (20) has a unique solution.

## 4. Conclusions

Many works in the literature and monographs have treated and developed mathematical models that appear in various real-world applications, for example, thermostats or heat controllers. One approach is to develop very complex iterations of popular models from real-world issues which can be described using inclusions or fractional differential equations [2,7,12,16].

In this work, we provide a comprehensive investigation of a class of hybrid fractional models of thermostats via nonlocal multi-valued boundary conditions (3) and (4) which satisfy multi-valued feedback control. The main tool of our study is applying Dhage's hybrid fixed point theorem [19]. The use of various approaches for certain differential and integral problems, including constraints or control variables, has recently been developed by several scholars, for example, in refs. [27–33]. This feedback control may be in an implicit form as in [27–30], multi-valued feedback control as in [32], or fractal feedback control [33].

We have established the continuous dependence of the unique solution of our problem on the control variable and on the set $S_\Phi$. In this study, we have investigated some qualitative properties of the solution of this problem, which encourages us to investigate and discuss additional singular dynamical systems that appear in a variety of natural and engineering phenomena.

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