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On the Global Nonexistence of a Solution for Wave Equations with Nonlinear Memory Term

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Abstract: The paper is devoted to the problem of the local existence for a solution to a nonlinear wave equation, with the dissipation given by a nonlinear form with the presence of a nonlinear memory term. Moreover, the global nonexistence of a solution is established using the test function method. We combine the Fourier transform and fractional derivative calculus to achieve our goal.

Keywords: Lyapunov functions; wave equations; well-posedness; nonlinear memory term; exponential; multiplier method; partial differential equations

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Citation: Reda, S.B.; Memou, A.; Berkane, A.; Himadan, A.; Moumen, A.; Saber, H.; Alraqad, T. On the Global Nonexistence of a Solution for Wave Equations with Nonlinear Memory Term. *Fractal Fract.* **2023**, *7*, 788. https://doi.org/10.3390/fractalfract7110788

Academic Editors: Arman Dabiri and Behrouz Parsa Moghaddam

Received: 20 September 2023
Revised: 19 October 2023
Accepted: 26 October 2023
Published: 29 October 2023

1. Introduction Setting

Many researchers investigate how the different terms that make up the PDE affect the existence of the solution and its stability and determine whether the solution exists in general with respect to time. The terms vary according to the physical phenomenon being studied and the modeled problem. The linear memory term is considered among the most famous terms, which effect studied problems. It often establishes the existence of the solution under some conditions. Researchers have recently turned to studying this term nonlinearly and choose the initial idea in a way that allows them to rewrite it in terms of fractional derivative.

To begin with, let $t \in (0, \infty), x \in \mathbb{R}^n, u = u(x,t)$. We look at the following Cauchy problem with a nonlinear memory term

$$\begin{cases} u_{tt} - \Delta_x u + (t+1)^r |u|^{m-1} u_t = \int\limits_0^t (t-s)^{-\varpi} |u|^p ds \\ u(t=0,x) = g(x), u_t(t=0,x) = h(x), \end{cases} \tag{1}$$

where u is the unknown real-valued function, $n > 0$, $m > 1$, $p > 1$, $r, \varpi \in \Omega = (0,1)$, and $g(x)$, $h(x)$ are the initial data. Many previous studies considered similar problems and obtained results indicating the existence of a solution at a specific point, followed by an explosion and nonexistence of the solution from a certain order, which is called the exponent. Here, we mention the most important works directly related to our model (1) containing the nonlinear memory term.

Taking $m = 1$ and $r = 0$ in (1), we obtain for $t \in (0, \infty), x \in \mathbb{R}^n$, the model

$$\begin{cases} u_{tt} - \Delta_x u + u_t = \int\limits_0^t (t-s)^{-\varpi} |u|^p ds \\ u(t=0,x) = g(x), u_t(t=0,x) = h(x) \ . \end{cases} \tag{2}$$

In [1], D'Abbicco showed the existence and the blow-up of the weak solution, and he proposed as a critical exponent

$$p(n,\varpi) = \max\left\{p_{\varpi}(n); \tfrac{1}{\varpi}\right\} \quad where \quad p_{\varpi}(n) = 1 + \tfrac{2(2-\varpi)}{(n-2(1-\varpi))_+},$$

where $(-2(1-\varpi)+n)_+ = \max\{(-2(1-\varpi)+n); 0\}$.

For $r = 0$ and $m > 1$, we obtain the model

$$\begin{cases} u_{tt} - \Delta_x u + |u|^{m-1} u_t = \int\limits_0^t (t-s)^{-\varpi} |u|^p ds \\ u(t=0,x) = g(x), u_t(t=0,x) = h(x). \end{cases} \tag{3}$$

A wave equation with structural damping and nonlinear memory was considered in [2]; the question of the existence of global solutions was proved, and in the subcritical case, the nonexistence of solutions for suitable arbitrarily small data was established. These two results improved the works [3,4], where a critical exponent was obtained.

In [5], the authors considered problem (1), where the coefficient of the frictional damping term is given by

$$a_0 = (1+|x|^2)^{\frac{-\alpha}{2}} (1+t)^{-\beta}, \tag{4}$$

in which $0 < a_0, 0 \le \beta, 1 > \alpha + \beta$. A blow-up result under certain positive data in $\mathbb{R}^n$ was obtained. Besides, the local existence in the energy space was also studied. In [6], Berbiche and Hakem showed the blow-up and local existence of a solution for this problem if $p > m > 1$. They proved that if h and g satisfy certain conditions and

$$n \le \min\left\{\tfrac{2(m+(1-\varpi)p)}{p-1+(1-\varpi)(m-1)}; \tfrac{2(1+(2-\varpi)p)}{\left(\frac{(p-1)(2-\varpi)}{p-m} - 1 + \varpi\right)(p-1)}\right\} \quad or \quad p < \frac{1}{\varpi},$$

then the weak solution does not exist.

For $m = 1$ and $r \in \Omega$, we have the model

$$\begin{cases} u_{tt} - \Delta_x u + (t+1)^r u_t = \int\limits_0^t (t-s)^{-\varpi} |u|^p ds \\ u(t=0,x) = g(x), u_t(t=0,x) = h(x). \end{cases}$$

In a similar study [7], it was proved that if g and h satisfy some conditions (we only take the case when $r \in \Omega$) and that

$$\frac{p}{p-1} > \inf_{d>0} \max\left\{ \frac{\frac{nd}{2}+1}{1-\varpi+d}; \sqrt{\frac{\frac{nd}{2}+1}{(1-\varpi)(1-r)} + \left(\frac{1-\varpi-(1-r)\frac{nd}{2}}{2(1-\varpi)(1-r)}\right)^2} - \frac{1-\varpi-(1-r)\frac{nd}{2}}{2(1-\varpi)(1-r)} \right\},$$

then a weak solution does not exist globally in time.

In [8], the authors considered a related problem "Semilinear wave equation with a nonlinear memory term" and studied the blow-up dynamic by using an iteration argument. The memory term used was the Riemann–Liouville fractional integral. In [9], Andrade and Tuan studied a problem named nonautonomous damped wave equation with a nonlinear memory term. The question of the well-posedness and spatial regularity of the problem was treated owing to the the theory of evolution process and sectorial operators. Models involving a linear kernel are not new and arise in heat conduction and linear viscoelasticity theory; we mention the works [10–17] and references therein.

In our model, we focus on (1) with $m > 1$ and $r \in \Omega$, thus extending the existing results. Also, we discuss the system in the unbounded domain $\mathbb{R}^n$, which makes the study more valuable from the application point of view. Toward this end, we used the method of Fourier transform combined with some techniques from fractional calculus. Mathematical models, consisting of nonlinear memory, are used to describe many phenomena in physics, chemistry, biology, and a number of other sciences. Such models arise in the study of wave processes in gas dynamics and nonlinear acoustics, in the description of waves in shallow water, hydromagnetic waves in cold plasma, ion-acoustic waves in cold plasma, electromagnetic waves in ferromagnets, and in a number of other applications. Thus, nonlinear wave equations play an important role in the study of wave processes in media with a nonlinear memory and make it possible to analytically describe these processes.

Mathematical models, consisting of generalized terms of damping and interaction between them, are used to describe many phenomena in various fields including physics, chemistry, and biology. Such models arise from many applications such as wave processes in gas dynamics and nonlinear acoustics, description of waves in shallow water, hydromagnetics waves in cold plasma, ion-acoustic waves in cold plasma, and electromagnetic waves in ferromagnetic. This is indeed our case; for researchers working in this field who want to learn something new and not easy, it is extremely an interesting section of modern science, including engineering and new physical principles.

The paper is organized as follows. In the next section, we list some preliminaries and the main problem. By using these tools, we prove the local existence to the system in Section 3. Finally, Section 4 is devoted to the blow-up results, see Theorem 8, which ensure the nonexistence of global solutions.

2. Preliminaries, Materials, and Methods

In this section, we recall some preliminary material, which is needed for this work.

Definition 1. *(Sobolev space) The (H^s) norm of a function $g \in (\mathbb{R}^n, \mathbb{R})$, denoted by $\|g\|_{H^s}$, is*

$$\|g\|_{H^s} = \left[\int_{\mathbb{R}} \left(1 + |\xi|^2\right)^s |\hat{g}|^2 d\xi\right]^{\frac{1}{2}},$$

where $\hat{g}$ is the well known Fourier transform variable of g. If $\|g\|_{H^s} < \infty$, then $g \in H^s$.

Definition 2. *The X^s norm of a function $u \in ([0,\infty) \times \mathbb{R}^n, \mathbb{R})$, denoted by $\|u\|_{X^s}$ is*

$$\|u\|_{X^s} = \sup_{0 \le t \le T} (\|u(t)\|_{H^s} + \|u_t(t)\|_{H^{s-1}}),$$

where $0 < T < \infty$. If $\|u\|_{X^s} < \infty$, then $u \in X^s$.

Theorem 1 ((Leibniz Integral Rule) [8])**.** *For $-\infty < a(x) < b(x) < +\infty$,*

$$\frac{d}{dx}\left(\int_{a(x)}^{b(x)} f dt\right) = f(b(x), x) \cdot \frac{d}{dx} b(x) - f(a(x), x) \cdot \frac{d}{dx} a(x).$$

Theorem 2 ([8])**.** *For any n-dimensional function $u(t_0) \in H^s$, if $s > n/2$, then there exists $C > 0$ so that*

$$\|u(t_0)\|_{L^\infty} \le C\|u(t_0)\|_{H^s}.$$

Theorem 3 ((Gronwall's Inequality) [18])**.** *Let $f(t)$ be a nonnegative and continuous function on $[0, T]$ satisfying, for all $0 \le t \le T$,*

$$f(t) \le \int_0^t f(s) ds.$$

Then, $f(t) = 0$ for all $0 \le t \le T$.

Theorem 4 ((Banach Contraction-Mapping Principle) [19]). *Let (X,d) be a complete metric space and $G : X \to X$ a map such that there exists $0 \leq \theta < 1$ satisfying $d(G(x), G(\xi)) \leq \theta d(x,\xi)$ for all $x, \xi \in X$. Then, there is a unique point $x_0 \in X$ such that $G(x_0) = x_0$.*

Lemma 1 ([19]). *For any $s \in (1,2)$ and $p \in (1,+\infty) \cap (s-1,+\infty)$, we have for a nonnegative function $f \in L^\infty(\mathbb{R}^n) \cap H^{s-1}(\mathbb{R}^n)$; then, $f^p \in H^{s-1}(\mathbb{R}^n)$, and there exists $C > 0$ such that*

$$\|f^p\|_{H^{s-1}(\mathbb{R}^n)} \leq C\|f\|^{p-1}_{L^\infty(\mathbb{R}^n)}\|u(t_0)\|_{H^{s-1}(\mathbb{R}^n)}.$$

We now state some results from fractional derivative calculus that will be used in the last section. As in [20], the operator $D^\alpha_{0|t}$ is a the fractional derivative operator of order α, defined by

$$D^\alpha_{0|t}u = d_t J^{1-\alpha}_{0|t}u,$$

and $J^{1-\alpha}_{0|t}$ represents the fractional integral of order $1-\alpha$, given by

$$J^{1-\alpha}_{0|t}u(t) = \frac{1}{\Gamma(1-\alpha)}\int\limits_0^t \frac{u(s)}{(t-s)^\alpha}ds \quad for \quad u \in C(\mathbb{R}).$$

Proposition 1 ([20]). *Let $g, h \in C[0,T]$. If $D^\alpha_{t|T}h(t)$ and $D^\alpha_{0|t}g(t)$ exist and belong to $C[0,T]$, then*

$$\int\limits_0^t g(s)D^\alpha_{s|T}h(s)ds = \int\limits_0^t h(s)D^\alpha_{0|s}g(s)ds,$$

for all $0 \leq t \leq T$.

Proposition 2 (see [19]). *For all $u \in L^q(0,T)$, $q \geq 1$, and $0 < t < T$, we have*

$$\left(D^\alpha_{0|t} \circ J^\alpha_{0|t}\right)(u) = u. \tag{5}$$

Moreover, for all $u \in C^l[0,T]$, $T > 0$, the following rule holds

$$(-1)^l d^l_t D^\alpha_{t|T}u(t) = D^{\alpha+l}_{t|T}u(t) \quad for \quad (l,\alpha) \in \mathbb{N} \times \Omega. \tag{6}$$

Corollary 1 ([21]). *If $\varphi(t) = \left(1 - \frac{t}{T}\right)^\beta_+$, $t \geq 0$, $T > 0$, $\beta \gg 1$, then*

$$D^{\alpha+i}_{t|T}\varphi(t) = CT^{-\beta}(T-t)^{\beta-\alpha-i}_+ \quad \alpha \in \Omega \quad i = 0,1,2, \tag{7}$$

$$\left(D^{\alpha+i}_{t|T}\varphi\right)(0) = CT^{-\alpha-i} \quad \alpha \in \Omega \quad i = 0,1,2. \tag{8}$$

Here, we demonstrate that the local solution to the problem (1) exists and is unique. To see that, let us first introduce the next Cauchy problem

$$\begin{cases} u_{tt} - \Delta_x u = F(u), \\ u(t=0,x) = g(x), u_t(t=0,x) = h(x). \end{cases} \tag{9}$$

Due to the nonlinearity of $F(u)$, it is difficult to provide a closed-form formula for the solution. However, under certain conditions one can show the existence of this solution.

2.1. Solving the Wave Equation

Let us first start by solving the wave equation with a source term, by using the Fourier Transform. Then, we deduce some properties and estimations satisfied by the solution to the problem (9).

Lemma 2 (The homogeneous case)**.** *If*

$$\hat{u}_0 = \begin{cases} \hat{g}\cos(|\xi|t) + \frac{\hat{h}}{|\xi|}\sin(|\xi|t) & |\xi| \neq 0 \\ \hat{h}t + \hat{g} & |\xi| = 0, \end{cases} \tag{10}$$

then u_0 solves the Equation (9) for $F \equiv 0$.

Proof. The Fourier transform of the homogeneous wave equation gives us

$$\hat{u}_{tt} + |\xi|^2 \hat{u} = 0. \tag{11}$$

This is an ordinary differential equation of second order, which has the characteristic equation

$$r^2 + |\xi|^2 = 0. \tag{12}$$

Thus, the solution can be written as

$$\hat{u} = A\cos(|\xi|t) + B\sin(|\xi|t).$$

Using the initial data, we obtain

$$\hat{u}(\xi, 0) = A\cos(|\xi|0) + B\sin(|\xi|0) = \hat{g},$$

yielding $A = \hat{g}$. Also, we have

$$\hat{u}_t(\xi, 0) = -|\xi|A\sin(|\xi|0) + |\xi|B\cos(|\xi|0) = \hat{h},$$

and so

$$|\xi|B = \hat{h}.$$

If $|\xi| \neq 0$, we obtain $B = \frac{\hat{h}}{|\xi|}$; hence,

$$\hat{u} = \hat{g}\cos(|\xi|t) + \frac{\hat{h}}{|\xi|}\sin(|\xi|t) \quad \text{for} \quad |\xi| \neq 0.$$

If $\xi = 0$, then the Equation (11) yields $\hat{u}_{tt} = 0$. Using the initial conditions, we deduce

$$\hat{u}_{tt} = \hat{h}t + \hat{g} \quad \text{for} \quad |\xi| = 0 \ .$$

□

Lemma 3 (Duhamels Principle)**.** *Suppose $w = \int_0^t v(x, t-s, s)ds$, where v solves*

$$\begin{cases} v_{tt} - \Delta_x v = 0 \\ v(t=0, x, s) = 0, v_t(t=0, x, s) = F(x, s); \end{cases}$$

then, w solves Equation (9) as the particular solution to $h \equiv g \equiv 0$.

Proof. By substituting w into Equation (9) and using the Leibniz integral Rule in Theorem 1, we obtain

$$\begin{aligned} w_{tt} - \Delta_x w &= \partial_t \left[v(t=0, x, s) + \int_0^t v(x, t-s, s)ds \right] - \int_0^t \Delta_x v(x, t-s, s)ds \\ &= v(x, t-s, s)|_{s=t} + \int_0^t [v_{tt}(x, t-s, s) - \Delta_x v(x, t-s, s)ds]ds \\ &= v(x, 0, s)| = F(x, t). \end{aligned}$$

□

Lemma 2, with $g \equiv 0$ and $h = F(x,s)$ gives, for v in Lemma 3, the following resuls

$$\hat{v}(\xi, t-s, s) = \begin{cases} \frac{\hat{F}(\xi,s)}{|\xi|}\sin(|\xi|(t-s)) & |\xi| \neq 0 \\ \hat{F}(\xi,s)(t-s) & |\xi| = 0. \end{cases}$$

By employing Lemma 3 and (11), we obtain, if we take u_0 as in Lemma 2 and w as in Lemma 3, that $u = u_0 + w$ solves Equation (9).

2.2. The Well-Posedness of the Wave Equation

In this subsection, we show some estimations for the solution, which are used later.

Theorem 5. *Let u be the solution to Equation (9). Then, the following inequality holds:*

$$\|u(t)\|_{H^s} < \|g\|_{H^s} + \|h\|_{H^{s-1}} + \int_0^t \|F(\tau)\|_{H^{s-1}} d\tau, \tag{13}$$

for all $t \in [0, +\infty)$.

Proof. We have from the (H^s) norm definition

$$\begin{aligned}
\|u(t)\|_{H^s} &= \left[\int_{\mathbb{R}} \left(1+|\xi|^2\right)^s |\hat{u}(t)|^2 d\xi\right]^{\frac{1}{2}} \\
&= \left[\int_{\xi\neq 0} \left(1+|\xi|^2\right)^s |\hat{u}(t)|^2 d\xi + \int_{\xi=0} \left(1+|\xi|^2\right)^s |\hat{u}(t)|^2 d\xi\right]^{\frac{1}{2}} \\
&= \left[\int_{\xi\neq 0} \left(1+|\xi|^2\right)^s \left|\hat{g}\cos(|\xi|t) + \frac{\hat{h}}{|\xi|}\sin(|\xi|t) + \int_0^t \frac{\hat{F}(\xi,\tau)}{|\xi|}\sin(|\xi|(t-\tau))d\tau\right|^2 d\xi\right]^{\frac{1}{2}} \\
&< \left[\int_{\xi\neq 0} \left(1+|\xi|^2\right)^s \left|\hat{g}\cos(|\xi|t) + \frac{\hat{h}}{|\xi|}\sin(|\xi|t)\right|^2 d\xi\right]^{\frac{1}{2}} \\
&+ \left[\int_{\xi\neq 0} \left(1+|\xi|^2\right)^s \left|\int_0^t \frac{\hat{F}(\xi,\tau)}{|\xi|}\sin(|\xi|(t-\tau))d\tau\right|^2 d\xi\right]^{\frac{1}{2}} \\
&< \left[\int_{\xi\neq 0} \left(1+|\xi|^2\right)^s |\hat{g}|^2 d\xi\right]^{\frac{1}{2}} + \left[\int_{\xi\neq 0} \left(1+|\xi|^2\right)^s \left|\hat{h}\right|^2 d\xi\right]^{\frac{1}{2}} \\
&+ \left[\int_{\xi\neq 0} \left(1+|\xi|^2\right)^s \int_0^t \left|\hat{F}(\xi,\tau)\right|^2 d\tau d\xi\right]^{\frac{1}{2}} \\
&< \|g\|_{H^s} + \left[\int_{\xi\neq 0} \left(1+|\xi|^2\right)^{s-1} \left(1+|\xi|^2\right) \left|\hat{h}\right|^2 d\xi\right]^{\frac{1}{2}} \\
&+ \left[\int_{\xi\neq 0} \left(1+|\xi|^2\right)^{s-1} \left(1+|\xi|^2\right) \int_0^t \left|\hat{F}(\xi,\tau)\right|^2 d\tau d\xi\right]^{\frac{1}{2}} \\
&< \|g\|_{H^s} + \|h\|_{H^{s-1}} + \int_0^t \|F(\tau)\|_{H^{s-1}} d\tau.
\end{aligned}$$

□

Theorem 6. *Let u be the solution to Equation (9). We have*

$$\|u_t(t)\|_{H^{s-1}} < \|g\|_{H^s} + \|h\|_{H^{s-1}} + \int_0^t \|F(\tau)\|_{H^{s-1}} d\tau, \tag{14}$$

for all $t \in [0, +\infty)$.

Proof. As in the above proof, we obtain

$$\begin{aligned}
&\|u_t(t)\|_{H^{s-1}} \\
&= \left[\int_{\xi\in\mathbb{R}} \left(1+|\xi|^2\right)^{s-1} |\hat{u}_t(t)|^2 d\xi\right]^{\frac{1}{2}} \\
&= \left[\int_{\xi\neq 0} \left(1+|\xi|^2\right)^{s-1} |\hat{u}_t(t)|^2 d\xi + \int_{\xi=0} \left(1+|\xi|^2\right)^{s-1} |\hat{u}_t(t)|^2 d\xi\right]^{\frac{1}{2}} \\
&= \left[\int_{\xi\neq 0} \left(1+|\xi|^2\right)^{s-1} \left| -|\xi|\hat{g}\sin(|\xi|t) + \hat{h}\cos(|\xi|t) + \int_0^t \hat{F}(\xi,\tau)\sin(|\xi|(t-\tau))d\tau \right|^2 d\xi\right]^{\frac{1}{2}} \\
&< \left[\int_{\xi\neq 0} \left(1+|\xi|^2\right)^{s-1} ||\xi|\hat{g}\sin(|\xi|t)|^2 d\xi\right]^{\frac{1}{2}} + \left[\int_{\xi\neq 0} \left(1+|\xi|^2\right)^{s-1} \left|\hat{h}\cos(|\xi|t)\right|^2 d\xi\right]^{\frac{1}{2}} \\
&+ \left[\int_{\xi\neq 0} \left(1+|\xi|^2\right)^{s-1} \int_0^t \left|\hat{F}(\xi,\tau)\sin(|\xi|(t-\tau))\right|^2 d\tau d\xi\right]^{\frac{1}{2}} \\
&< \left[\int_{\xi\neq 0} \left(1+|\xi|^2\right)^{s} \frac{|\xi|^2}{1+|\xi|^2} |\hat{g}|^2 d\xi\right]^{\frac{1}{2}} + \left[\int_{\xi\neq 0} \left(1+|\xi|^2\right)^{s-1} \left|\hat{h}\right|^2 d\xi\right]^{\frac{1}{2}} \\
&+ \left[\int_{\xi\neq 0} \left(1+|\xi|^2\right)^{s-1} \int_0^t \left|\hat{F}(\xi,\tau)\right|^2 d\tau d\xi\right]^{\frac{1}{2}} \\
&< \|g\|_{H^s} + \|h\|_{H^{s-1}} + \int_0^t \|F(\tau)\|_{H^{s-1}} d\tau.
\end{aligned}$$

□

Corollary 2. *For the solution to Equation (9), we have*

$$\|u\|_{X^s} < \|g\|_{H^s} + \|h\|_{H^{s-1}} + \int_0^T \|F(\tau)\|_{H^{s-1}} d\tau. \tag{15}$$

Proof. We have $\|u\|_{X^s} = \sup_{0\le t\le T} \left(\|u(t)\|_{H^s} + \|u_t(t)\|_{H^{s-1}}\right)$. Thus, the result follows from Theorems 5 and 6. □

3. First Main Result: Local Existence

Now, we are ready to prove the existence and the uniqueness result of the local solution to our problem.

Theorem 7. *Let $n > 0$, $s > \frac{n}{2} + 1$, $r \in (0, 1)$, and $m, p \in (1, +\infty) \cap (s-1, +\infty)$. Then, for $g \in H^s(\mathbb{R}^n)$ and $h \in H^{s-1}(\mathbb{R}^n)$, problem (1) has a unique solution*

$$u \in C([0, T]; H^s(\mathbb{R}^n)) \cap C^1\left([0, T]; H^{s-1}(\mathbb{R}^n)\right),$$

with the positive number T only depending on $\|g\|_{H^s} + \|h\|_{H^{s-1}}$.

Proof. The next relation $m|u|^{m-1}u_t = \partial_t\left(|u|^{m-1}u\right)$ leads us to introduce a new unknown v satisfying $u = v_t$. Then, we obtain a new problem with a locally Lipschitz nonlinear memory term

$$\begin{cases} v_{tt} - \Delta_x v = -\frac{b(t)}{m}|v_t|^{m-1}v_t + \frac{1}{1-\varpi}\int\limits_0^t (t-s)^{1-\varpi}|v_t(s,x)|^p ds \\ +\frac{1}{m}\int\limits_0^t b'(t)|v_t|^{m-1}v_t ds + \frac{1}{m}|g|^{m-1}g(x) + h(x), \end{cases} \tag{16}$$

where $v(t=0,x) = 0, v_t(t=0,x) = g(x)$, and it is easy to check that $u \in C([0,T];H^s(\mathbb{R}^n))$ $\cap\, C^1\left([0,T];H^{s-1}(\mathbb{R}^n)\right)$ is the solution to (1), if and only if v is a solution (16) in the class

$$\begin{cases} v \in C([0,T];H^s(\mathbb{R}^n)) \\ v_t \in C^1\left([0,T];H^{s-1}(\mathbb{R}^n)\right), v_{tt} \in C^1\left([0,T];H^{s-1}(\mathbb{R}^n)\right). \end{cases}$$

Let us define

$$\begin{aligned} &X_T = C([0,T];H^s(\mathbb{R}^n)) \cap C^1\left([0,T];H^{s-1}(\mathbb{R}^n)\right) \\ &Y_T = L^\infty([0,T];H^s(\mathbb{R}^n)) \cap W^{1,\infty}\left([0,T];H^{s-1}(\mathbb{R}^n)\right) \\ &B_{T,M} = \left\{u \in Y_T;\ \sup_{0\le t\le T}\left(\|u(t,.)\|_{H^s} + \|u_t(t,.)\|_{H^{s-1}}\right) \le M\right\}. \end{aligned}$$

Now, we set that $X_{T,M} = B_{T,M} \cap X_T$, since $X_T \subset Y_T$ and $X_{T,M} \subset B_{T,M}$. Set

$$\begin{aligned} F(v_t) = &-\frac{b(t)}{m}|v_t|^{m-1}v_t + \frac{1}{1-\varpi}\int\limits_0^t (t-s)^{1-\varpi}|v_t(s,x)|^p ds \\ &+\frac{1}{m}\int\limits_0^t b'(t)|v_t|^{m-1}v_t ds + \frac{1}{m}|g|^{m-1}g(x) + h(x). \end{aligned} \tag{17}$$

Now, we introduce for any $w \in Y_T$ the map $\Psi[w] = v$, where $v \in X_T$ is a solution to

$$\begin{cases} v_{tt} - \Delta_x v = F(w_t) \quad (0,T)\times\mathbb{R}^n \\ v(t=0,x) = 0, v_t(t=0,x) = g(x) \quad x \in \mathbb{R}^n. \end{cases} \tag{18}$$

Our goal is to show that there exists a unique v such that $\Psi[v] = v$. For that, we start by showing Ψ is well defined. Indeed, let $w \in Y_{T,M}$ then $\Psi[w] \in X_{T,M}$ for sufficiently small $T > 0$. We have from the Corollary 2

$$\|v\|_{H^s} + \|v_t\|_{H^{s-1}} < \|g\|_{H^{s-1}} + \int\limits_0^T \|F(\tau)\|_{H^{s-1}}d\tau, \tag{19}$$

and from Lemma 1, since $s > \frac{n}{2} + 1$, we obtain

$$
\begin{aligned}
&\int_0^t \|F(\tau,.)\|_{H^{s-1}} d\tau \\
&\leq \frac{b(t)}{m} \int_0^t \left\| |w_t|^{m-1} w_t(\tau,x) \right\|_{H^{s-1}} d\tau + \frac{1}{1-\varpi} \int_0^t \int_0^t (\tau-s)^{1-\varpi} \left\| |w_t(s,x)|^p \right\|_{H^{s-1}}(s) d\tau ds \\
&+ \frac{T}{m} \left\| |g|^{m-1} g(x) + h(x) \right\|_{H^{s-1}} + \frac{1}{m} \int_0^t \int_0^\tau b'(t) \left\| |w_t|^{m-1} w_t \right\|_{H^{s-1}} d\tau ds \\
&< T \|w_t\|_{L^\infty}^{m-1} \sup_{0\leq t\leq T} \|w_t\|_{H^{s-1}} + \frac{1}{1-\varpi} \int_0^t (\tau-s)^{1-\varpi} \|w_t\|_{L^\infty}^{p-1} \sup_{0\leq t\leq T} \|w_t\|_{H^{s-1}} ds \\
&+ T\left(\|g\|_{H^s}^m + \|h\|_{H^{s-1}}\right) + \frac{1}{m} \int_0^t \int_0^\tau b'(t) \|w_t\|_{L^\infty}^{m-1} \sup_{0\leq t\leq T} \|w_t\|_{H^{s-1}} d\tau ds.
\end{aligned} \tag{20}
$$

Using the Sobolev embedding in Theorem 2, we find

$$
\begin{aligned}
\int_0^t \|F(\tau,.)\|_{H^{s-1}} d\tau < T \sup_{0\leq t\leq T} \|w_t\|_{H^{s-1}}^m + \sup_{0\leq t\leq T} \|w_t\|_{H^{s-1}}^p \int_0^t (\tau-s)^{1-\varpi} ds \\
+ T\left(\|g\|_{H^s}^m + \|h\|_{H^{s-1}}\right) + \sup_{0\leq t\leq T} \|w_t\|_{H^{s-1}}^m \int_0^t b'(t) dt.
\end{aligned}
$$

Now, let $M = 4\left(\|g\|_{H^s}^m + \|h\|_{H^{s-1}}\right)$; then, we can conclude that

$$
\int_0^t \|F(\tau,.)\|_{H^{s-1}} d\tau \leq C\Big(TM^m + T^{2-\varpi}M^p + (1+T)^r M^m + TM,\Big), \tag{21}
$$

and so

$$
\begin{aligned}
\sup_{0\leq t\leq T} \left(\|v\|_{H^s} + \|v_t\|_{H^{s-1}}\right) &\leq C\Big(\|g\|_{H^s} + TM^m + T^{2-\varpi}M^p + (1+T)^r M^m + TM\Big) \\
&\leq C\Big(M + TM^m + T^{2-\varpi}M^p + (1+T)^r M^m + TM.\Big).
\end{aligned}
$$

Therefore, we have

$$
\sup_{0\leq t\leq T} \left(\|v\|_{H^s} + \|v_t\|_{H^{s-1}}\right) \leq C_{T,M} M,
$$

where

$$
C_{T,M} = C\left(\frac{1}{4} + TM^{m-1} + T^{2-\varpi}M^{p-1} + (1+T)^r M^{m-1} + \frac{T}{4}\right).
$$

Now, for a sufficiently small T, we can choose $T_1 > 0$ such that $C_{T,M} \leq 1$ for any $T \in (0, T_1]$. Hence, $\Psi[w_t] \in X_{T,M}$, and thus, Ψ is well defined.

Next, we show that Ψ is a contraction mapping in $X_{T,M}$. For that, let $w_1, w_2 \in Y_{T,M}$. Then, $\Psi[w_1], \Psi[w_2] \in X_{T,M}$. Let v_1, v_2 be two solutions to Equation (18). Set $\tilde{v} = v_1 - v_2$. Then, $\tilde{v}$ satisfies

$$\begin{cases} \tilde{v}_{tt} - \Delta_x \tilde{v} = \frac{1}{1-\varpi} \int\limits_0^t (t-\tau)^{1-\varpi} \left(|(w_1)_t(\tau,x)|^p - |(w_2)_t(\tau,x)|^p \right) d\tau \\ + \frac{1}{m} \int\limits_0^t b'(s) \left[|(w_1)_t(s,x)|^{m-1} (w_1)_t(s,x) - |(w_2)_t(s,x)|^{m-1} (w_2)_t(s,x) \right] ds \\ + \frac{b(t)}{m} \left[|(w_1)_t|^{m-1} (w_1)_t - |(w_2)_t|^{m-1} (w_2)_t t, x) \right] \\ \tilde{v}(t=0,x) = \tilde{v}_t(t=0,x) = 0 \quad x \in \mathbb{R}^n. \end{cases} \tag{22}$$

Since $w_i \in Y_{T,M}$, by Sobolev's embedding, we conclude that

$$v_i in Y_{T,M} \quad and \quad \tilde{v} \in C([0,T]; H^s(R^n)) \cap C^1\left([0,T]; H^{s-1}(R^n)\right).$$

Therefore,

$$\begin{aligned} &\|\tilde{v}(t,.)\|_{H^s} + \|\tilde{v}_t(t,.)\|_{H^{s-1}} \\ &< \int\limits_0^t \int\limits_0^s (s-\tau)^{1-\varpi} \left\| \left(|(w_1)_t(\tau,x)|^p - |(w_2)_t(\tau,x)|^p \right) \right\|_{H^{s-1}} d\tau ds \\ &+ \frac{1}{m} \int\limits_0^t \int\limits_0^s b'(s) \left\| \left[|(w_1)_t(s,x)|^{m-1} (w_1)_t(s,x) - |(w_2)_t(s,x)|^{m-1} (w_2)_t(s,x) \right] \right\|_{H^{s-1}} ds dt \\ &+ \frac{1}{m} \int\limits_0^t b(s) \left\| \left[|(w_1)_t(s,x)|^{m-1} (w_1)_t(s,x) - |(w_2)_t(s,x)|^{m-1} (w_2)_t(s,x) \right] \right\|_{H^{s-1}} ds dt. \end{aligned} \tag{23}$$

Since $|v|^{l-1} v$ is a C^1 function with $l > 0$, the mean value theorem leads to

$$\begin{vmatrix} \left| |v_1|^{l-1} v_1 - |v_2|^{l-1} v_2 \right| \le C\left(|v_1|^{l-1} + |v_2|^{l-1} \right) |v_1 - v_2| \\ \left| |v_1|^{l} - |v_2|^{l} \right| \le C\left(|v_1|^{l-1} + |v_2|^{l-1} \right) |v_1 - v_2|. \end{vmatrix}$$

From Sobolev's inequality $\|\tilde{v}\|_{L^\infty} \le C \|\tilde{v}\|_{H^{s-1}}$, it follows that for $s > \frac{n}{2} + 1$,

$$\begin{aligned} &\|\tilde{v}\|_{H^s} + \|\tilde{v}_t\|_{H^{s-1}} \\ &< \int\limits_0^t (t-\tau)^{1-\varpi} \left[\| |(w_1)_t(\tau,x)| \|_{H^{s-1}}^{p-1} + \| |(w_2)_t(\tau,x)| \|_{H^{s-1}}^{p-1} \right] \| |(w_1 - w_2)_t(\tau,x)| \|_{H^{s-1}} d\tau \\ &+ \frac{1}{m} \int\limits_0^t b'(\tau) \left[\| |(w_1)_t(\tau,x)| \|_{H^{s-1}}^{m-1} + \| |(w_2)_t(\tau,x)| \|_{H^{s-1}}^{m-1} \right] \| |(w_1 - w_2)_t(\tau,x)| \|_{H^{s-1}} d\tau \\ &+ \frac{1}{m} \int\limits_0^t b'(\tau) \left[\| |(w_1)_t(\tau,x)| \|_{H^{s-1}}^{m-1} + \| |(w_2)_t(\tau,x)| \|_{H^{s-1}}^{m-1} \right] \| |(w_1 - w_2)_t(\tau,x)| \|_{H^{s-1}} d\tau \\ &< \left(T^{2-\varpi} M^{p-1} + (1+T)^{r+1} M^{m-1} \right) \cdot \sup_{0 \le t \le T} \| |(w_1 - w_2)_t(t,x)| \|_{H^{s-1}}. \end{aligned} \tag{24}$$

So, there exists a positive C such that

$$\begin{aligned} &\| [\Psi[w_1] - \Psi[w_2]](t,.) \|_{H^{s-1}} \\ &\le C\left(T^{2-\varpi} M^{p-1} + (1+T)^{r+1} M^{m-1} \right) \sup_{0 \le t \le T} \| |(w_1 - w_2)_t| \|_{H^{s-1}}. \end{aligned}$$

Fix $T \in (0, T_1]$ small enough such that $C\left(T^{2-\varpi} M^{p-1} + (1+T)^{r+1} M^{m-1}\right) < \frac{1}{2}$. Hence,

$$\|\Psi[w_1] - \Psi[w_2]\|_{X_{T,M}} \leq \frac{1}{2} \|(w_1 - w_2)(t,.)\|_{X_{T,M}}.$$

Therefore, owing to Banach's fixed point principle, one can directly conclude the proof of the unique local solution. □

4. Second Main Result: Blowing-Up

Here, the blow-up of the solution to system (1) is established. We start by giving the following definition regarding the meaning of the solution to (1).

Definition 3. *Let $T > 0$, $r \in (0,1)$, $0 < \varpi < 1$, $\alpha = 1 - \varpi$, $b(t) = (t+1)^r$, and $g \in L^1_{loc}(\mathbb{R}^n) \cap L^m_{loc}(\mathbb{R}^n)$, $h \in L^1_{loc}(\mathbb{R}^n)$. We call u a weak solution, if $u \in L^p\left((0,T), L^p_{loc}(\mathbb{R}^n)\right) \cap L^m\left((0,T), L^m_{loc}(\mathbb{R}^n)\right)$, and it satisfies*

$$\begin{aligned}
&\Gamma(\alpha) \int_0^T \int_{\mathbb{R}^n} J^\alpha_{0|t}(|u|^p)\varphi dx dt + \int_{\mathbb{R}^n} h(x)\varphi(0,x)dx \\
&- \int_{\mathbb{R}^n} g(x)\varphi_t(0,x)dx + \frac{1}{m}\int_{\mathbb{R}^n} |g|^{m-1} g\varphi(0,x)dx \\
&= \int_0^T \int_{\mathbb{R}^n} u\varphi_{tt} dx dt - \frac{1}{m}\int_0^T \int_{\mathbb{R}^n} b'(t)|u|^{m-1} u\varphi dx dt \\
&- \frac{1}{m}\int_0^T \int_{\mathbb{R}^n} b(t)|u|^{m-1} u\varphi_t dx dt - \int_0^T \int_{\mathbb{R}^n} u\Delta_x\varphi dx dt,
\end{aligned} \tag{25}$$

for nonnegative functions (test)

$$\varphi \in C^2([0,T] \times \mathbb{R}^n),$$

where

$$\varphi(T,x) = \varphi_t(T,x) = 0.$$

In the sequel, we have $g > f$ and $\exists C > 0$, where $f \leq Cg$, and the nonnegative constant C is supposed to be independent of $T > 0$.

Theorem 8. *Let $n > 0$ and $0 < \varpi < 1$, and let $m > p > 1$. Under the following conditions on the functions (g,h)*

$$\begin{aligned}
&\int_{\mathbb{R}^n} g(x)dx > 0 \\
&\int_{\mathbb{R}^n} h(x)dx > 0 \\
&\int_{\mathbb{R}^n} |g(x)|^{m-1} g(x)dx > 0,
\end{aligned} \tag{26}$$

and if

$$\frac{p}{p-1} > \inf_{d>0} \max\left\{\frac{\frac{nd}{2}+1}{\alpha+d}, \sqrt{\left(\frac{B}{2A}\right)^2 - \frac{C}{A}} - \frac{B}{2A}\right\}, \tag{27}$$

with

$$\begin{aligned}
A &= (1-m)(r-2+m)(1+\rho) - \alpha(r-2+m) - (1-r)(1-m), \\
B &= m[(2m-3+r)(1+\rho) + \alpha + (1-r)], \\
C &= -m^2(1+\rho),
\end{aligned}$$

we have the weak solution in the sense of definition (3) that (1) does not exist globally in t.

Proof. Seeking a contradiction, assume that u is a nontrivial weak solution to the problem (1), which exists globally in time. We apply the test function method. For some $T>0$, we choose the test function as follows

$$\varphi = D^{\alpha}_{t|T}\Psi = \varphi_1^l(x) D^{\alpha}_{t|T}\varphi_2(t), \tag{28}$$

where $\varphi_1(x) = \Phi\left(\frac{|x|^2}{\eta}\right)$ and $\varphi_2(t) = \left(1-\frac{t}{T}\right)_+^{\beta}$ with $1 \leq \beta$ and $0 < \eta$, $\Phi \in C^{\infty}(\mathbb{R}_+)$ is a cutoff nonincreasing function satisfying

$$\Phi(z) = \begin{cases} 1\,, & 0 \leq z \leq 1 \\ 0\,, & 2 \leq z, \end{cases}$$

with $0 \leq \Phi \leq 1$, and for all $z \in \mathbb{R}$, we have $|\Phi'(z)| \leq \frac{c}{1+z}$ for $c>0$. Now, we rewrite the problem of the weak solution as

$$\begin{aligned}
&\Gamma(\alpha)\int_0^T\int_{\mathbb{R}^n} J^{\alpha}_{0|t}\big(|u|^p\big) D^{\alpha}_{t|T}\Psi dxdt + \int_{\mathbb{R}^n} h(x) D^{\alpha}_{t|T}\Psi(0,x)dx \\
&- \int_{\mathbb{R}^n} g(x)\partial_t D^{\alpha}_{t|T}\Psi(0,x)dx + \frac{1}{m}\int_{\mathbb{R}^n} b(0)|g|^{m-1} g\Psi(0,x)dx \\
&= \int_0^T\int_{\mathbb{R}^n} u\partial_t^2 D^{\alpha}_{t|T}\Psi dxdt - \frac{1}{m}\int_0^T\int_{\mathbb{R}^n} b'(t)|u|^{m-1} u D^{\alpha}_{t|T}\Psi dxdt \\
&- \frac{1}{m}\int_0^T\int_{\mathbb{R}^n} b(t)|u|^{m-1} u\partial_t D^{\alpha}_{t|T}\Psi dxdt - \int_0^T\int_{\mathbb{R}^n} u\Delta_x D^{\alpha}_{t|T}\Psi dxdt.
\end{aligned} \tag{29}$$

By using the identity (6) in Proposition 2, we obtain

$$\begin{aligned}
&\Gamma(\alpha)\int_0^T\int_{\mathbb{R}^n} J^{\alpha}_{0|t}\big(|u|^p\big) D^{\alpha}_{t|T}\Psi dxdt + \int_{\mathbb{R}^n} h(x)\varphi_1^l(x) D^{\alpha}_{t|T}\varphi_2(0)dx \\
&- \int_{\mathbb{R}^n} g(x)\varphi_1^l(x) D^{\alpha+1}_{t|T}\varphi_2(0)dx + \frac{1}{m}\int_{\mathbb{R}^n} b(0)|g|^{m-1} g\varphi_1^l(x) D^{\alpha}_{t|T}\varphi_2(0)dx \\
&= \int_0^T\int_{\mathbb{R}^n} u\varphi_1^l(x) D^{\alpha+2}_{t|T}\varphi_2(t)dxdt - \frac{1}{m}\int_0^T\int_{\mathbb{R}^n} b'(t)|u|^{m-1} u\varphi_1^l(x) D^{\alpha}_{t|T}\varphi_2(t)dxdt \\
&- \frac{1}{m}\int_0^T\int_{\mathbb{R}^n} b(t)|u|^{m-1} u\varphi_1^l(x) D^{\alpha+1}_{t|T}\varphi_2(t)dxdt - \int_0^T\int_{\mathbb{R}^n} u\Delta_x\varphi_1^l(x) D^{\alpha}_{t|T}\varphi_2(t)dxdt.
\end{aligned} \tag{30}$$

Applying Corollary 1 yields

$$\begin{aligned}
&\Gamma(\alpha)\int_0^T\int_{\mathbb{R}^n} J^{\alpha}_{0|t}\big(|u|^p\big) D^{\alpha}_{t|T}\Psi dxdt + CT^{-\alpha}\int_{\mathbb{R}^n} h(x)\varphi_1^l(x)dx \\
&+ CT^{-\alpha-1}\int_{\mathbb{R}^n} g(x)\varphi_1^l(x)dx + CT^{-\alpha}\int_{\mathbb{R}^n} |g|^{m-1} g(x)\varphi_1^l(x)dx \\
&= \int_0^T\int_{\mathbb{R}^n} u\varphi_1^l(x) D^{\alpha+2}_{t|T}\varphi_2(t)dxdt - \frac{1}{m}\int_0^T\int_{\mathbb{R}^n} b'(t)|u|^{m-1} u\varphi_1^l(x) D^{\alpha}_{t|T}\varphi_2(t)dxdt \\
&- \frac{1}{m}\int_0^T\int_{\mathbb{R}^n} b(t)|u|^{m-1} u\varphi_1^l(x) D^{\alpha+1}_{t|T}\varphi_2(t)dxdt - \int_0^T\int_{\mathbb{R}^n} u\Delta_x\varphi_1^l(x) D^{\alpha}_{t|T}\varphi_2(t)dxdt.
\end{aligned} \tag{31}$$

To simplify the first term of the LHS of (31), we integrate by parts with the use of Proposition 1 and (5). Thus, we obtain

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^n} J_{0|t}^{\alpha}\left(|u|^p\right)D_{t|T}^{\alpha}\Psi dxdt &= \int_0^T\int_{\mathbb{R}^n} D_{0|T}^{\alpha}\left(J_{0|t}^{\alpha}\left(|u|^p\right)\right)\Psi dxdt \\ &= \int_0^T\int_{\mathbb{R}^n} |u|^p\Psi dxdt.\end{aligned} \tag{32}$$

Regarding the other terms, we denote by Ω_T the support of φ_1 as

$$\Omega_T = \sup\left\{x \in R^n : |x|^2 \le 2T\right\}.$$

Now, using the Lebesgue dominated convergence theorem, we obtain

$$\begin{aligned}&\lim_{T\to\infty}\int_{\Omega_T} g(x)\varphi_1^l(x)dx = \int_{\mathbb{R}^n} g(x)dx\\ &\lim_{T\to\infty}\int_{\Omega_T} |g(x)|^{m-1}g(x)\varphi_1^l(x)dx = \int_{\mathbb{R}^n} |g(x)|^{m-1}g(x)dx\\ &\lim_{T\to\infty}\int_{\Omega_T} h(x)\varphi_1^l(x)dx = \int_{\mathbb{R}^n} h(x)dx.\end{aligned}$$

We also have

$$\begin{aligned}&\lim_{T\to\infty} g(x)\varphi_1^l(x) = g(x)\\ &\lim_{T\to\infty} h(x)\varphi_1^l(x) = h(x)\\ &\lim_{T\to\infty} |g(x)|^{m-1}g(x)\varphi_1^l(x) = |g(x)|^{m-1}g(x).\end{aligned}$$

By assumption, we have $(g,h) \in L^m_{loc}(R^n)\times L^1_{loc}(R^n)$ satisfying

$$\begin{aligned}&\int_{\mathbb{R}^n} g(x)dx > 0\\ &\int_{\mathbb{R}^n} h(x)dx > 0\\ &\int_{\mathbb{R}^n} |g(x)|^{m-1}g(x)dx > 0.\end{aligned}$$

This implies that

$$\begin{aligned}&\int_{\mathbb{R}^n} g(x)\varphi_1^l(x)dx > 0\\ &\int_{\mathbb{R}^n} h(x)\varphi_1^l(x)dx > 0\\ &\int_{\mathbb{R}^n} |g(x)|^{m-1}g(x)\varphi_1^l(x)dx > 0,\end{aligned}$$

for $T > 0$. Now, we deal with the RHS of (31) of the weak formulation of solutions. Using the identity

$$\Delta_x\varphi_1^l = l\varphi_1^{l-1}\Delta_x\varphi_1 + l(l-1)\varphi_1^{l-2}|\nabla_x\varphi_1|^2,$$

and the fact that $0 \le \varphi_1 \le 1$, we obtain

$$\left|l\varphi_1^{l-1}\Delta_x\varphi_1 + l(l-1)\varphi_1^{l-2}|\nabla_x\varphi_1|^2\right| \le C\varphi_1^{l-2}\left(|\Delta_x\varphi_1| + |\nabla_x\varphi_1|^2\right).$$

Thus,

$$\int_0^T\int_{R^n}\left|u\Delta_x\varphi_1^l(x)D_{t|T}^{\alpha}\varphi_2(t)\right|dxdt < \int_0^T\int_{R^n} |u|\varphi_1^{l-2}\left(|\Delta_x\varphi_1| + |\nabla_x\varphi_1|^2\right)D_{t|T}^{\alpha}\varphi_2(t)dxdt.$$

By applying the triangular inequality, we write the RHS of (31) of the weak formulation of solutions as

$$\begin{aligned}\Gamma(\alpha)\int_0^T\int_{\mathbb{R}^n}|u|^p\Psi dxdt &< \int_0^T\int_{\mathbb{R}^n}\left|u\varphi_1^l(x)D_{t|T}^{\alpha+2}\varphi_2(t)\right|dxdt \\ &+\int_0^T\int_{\mathbb{R}^n}|b'(t)||u|^m\varphi_1^l(x)\left|D_{t|T}^{\alpha}\varphi_2(t)\right|dxdt \\ &+\int_0^T\int_{\mathbb{R}^n}|b(t)||u|^m\varphi_1^l(x)\left|D_{t|T}^{\alpha+1}\varphi_2(t)\right|dxdt \\ &+\int_0^T\int_{\mathbb{R}^n}|u|\varphi_1^{l-2}\Big(|\Delta_x\varphi_1|+|\nabla_x\varphi_1|^2\Big)D_{t|T}^{\alpha}\varphi_2(t)dxdt.\end{aligned}\tag{33}$$

The ε-Young's inequality states that, for all $A, B, p, q > 0$,

$$AB \le \varepsilon A^p + C(\varepsilon)B^q \quad pq = q + p.$$

Now, applying ε-Young's inequality on the terms of the RHS of (33) yields

$$\begin{aligned}&\int_0^T\int_{\mathbb{R}^n}|u|\varphi_1^l(x)\left|D_{t|T}^{\alpha+2}\varphi_2(t)\right|dxdt \\ &=\int_0^T\int_{\mathbb{R}^n}|u|\Psi^{\frac{1}{p}}\Psi^{-\frac{1}{p}}\varphi_1^l(x)\left|D_{t|T}^{\alpha+2}\varphi_2(t)\right|dxdt \\ &< \varepsilon\int_0^T\int_{\mathbb{R}^n}|u|^p\Psi dxdt + C(\varepsilon)\int_0^T\int_{\mathbb{R}^n}\varphi_1^l(x)\varphi_2^{-\frac{1}{p-1}}(t)\left|D_{t|T}^{\alpha+2}\varphi_2(t)\right|^{\frac{p}{p-1}}dxdt,\end{aligned}\tag{34}$$

and

$$\begin{aligned}&\int_0^T\int_{\mathbb{R}^n}|b'(t)||u|^m\varphi_1^l(x)\left|D_{t|T}^{\alpha}\varphi_2(t)\right|dxdt \\ &=\int_0^T\int_{\mathbb{R}^n}|u|^m\Psi^{\frac{m}{p}}\Psi^{-\frac{m}{p}}|b'(t)|\varphi_1^l(x)\left|D_{t|T}^{\alpha}\varphi_2(t)\right|dxdt \\ &< \varepsilon\int_0^T\int_{\mathbb{R}^n}|u|^p\Psi dxdt + C(\varepsilon)\int_0^T\int_{\mathbb{R}^n}\varphi_1^l(x)\varphi_2^{-\frac{m}{p-m}}(t)|b'(t)|^{\frac{p}{p-m}}\left|D_{t|T}^{\alpha}\varphi_2(t)\right|^{\frac{p}{p-m}}dxdt;\end{aligned}\tag{35}$$

then,

$$\begin{aligned}&\int_0^T\int_{\mathbb{R}^n}|b(t)||u|^m\varphi_1^l(x)\left|D_{t|T}^{\alpha+1}\varphi_2(t)\right|dxdt \\ &=\int_0^T\int_{\mathbb{R}^n}|u|^m\Psi^{\frac{m}{p}}\Psi^{-\frac{m}{p}}|b(t)|\varphi_1^l(x)\left|D_{t|T}^{\alpha+1}\varphi_2(t)\right|dxdt \\ &< \varepsilon\int_0^T\int_{\mathbb{R}^n}|u|^p\Psi dxdt + C(\varepsilon)\int_0^T\int_{\mathbb{R}^n}\varphi_1^l(x)\varphi_2^{-\frac{m}{p-m}}(t)|b(t)|^{\frac{p}{p-m}}\left|D_{t|T}^{\alpha+1}\varphi_2(t)\right|^{\frac{p}{p-m}}dxdt,\end{aligned}\tag{36}$$

and

$$\int_0^T\int_{\mathbb{R}^n} |u|\varphi_1^{l-2}(x)\Big(|\Delta_x\varphi_1(x)|+|\nabla_x\varphi_1(x)|^2\Big)D^{\alpha}_{t|T}\varphi_2(t)dxdt<\varepsilon\int_0^T\int_{\mathbb{R}^n}|u|^p\Psi dxdt$$
$$+C(\varepsilon)\int_0^T\int_{\mathbb{R}^n}\varphi_1^{l-\frac{2p}{p-1}}(x)\Big(|\Delta_x\varphi_1(x)|^{\frac{p}{p-1}}+|\nabla_x\varphi_1(x)|^{2\frac{p}{p-1}}\Big)\varphi_2^{-\frac{1}{p-1}}(t)\Big|D^{\alpha}_{t|T}\varphi_2(t)\Big|^{\frac{p}{p-1}}dxdt. \tag{37}$$

By choosing an estimation for ε small enough, from (34)–(37) with (33), we obtain

$$\int_0^T\int_{R^n}|u|^p\Psi dxdt<J_1+J_2+J_3+J_4, \tag{38}$$

with

$$\begin{aligned}
J_1&=\int_0^T\int_{\mathbb{R}^n}\varphi_1^l(x)\varphi_2^{-\frac{1}{p-1}}(t)\Big|D^{\alpha+2}_{t|T}\varphi_2(t)\Big|^{\frac{p}{p-1}}dxdt,\\
J_2&=\int_0^T\int_{\mathbb{R}^n}\varphi_1^l(x)\varphi_2^{-\frac{m}{p-m}}(t)|b'(t)|^{\frac{p}{p-m}}\Big|D^{\alpha}_{t|T}\varphi_2(t)\Big|^{\frac{p}{p-m}}dxdt,\\
J_3&=\int_0^T\int_{\mathbb{R}^n}\varphi_1^l(x)\varphi_2^{-\frac{m}{p-m}}(t)|b(t)|^{\frac{p}{p-m}}\Big|D^{\alpha+1}_{t|T}\varphi_2(t)\Big|^{\frac{p}{p-m}}dxdt,\\
J_4&=\int_0^T\int_{\mathbb{R}^n}\varphi_1^{l-2q}(x)\Big(|\Delta_x\varphi_1(x)|^q+|\nabla_x\varphi_1(x)|^{2q}\Big)\varphi_2^{-\frac{1}{p-1}}(t)\Big|D^{\alpha}_{t|T}\varphi_2(t)\Big|^q dxdt.
\end{aligned}$$

To estimate the integrals J_1, J_2, J_3, and J_4, we use the change in the variables $\xi=T^{-\frac{d}{2}}$, $s=T^{-1}t$ with ($\eta=T^{\frac{d}{2}}$), and d is a positive constant (chosen later). Note that $supp\varphi_1=\Omega_{T^d}$. Thus, by Fubini's Theorem, we obtain

$$J_1=\left(\int_{\Omega_{T^d}}\varphi_1^l(x)dx\right)\left(\int_0^T\varphi_2^{-\frac{1}{p-1}}(t)\Big|D^{\alpha+2}_{t|T}\varphi_2(t)\Big|^{\frac{p}{p-1}}dt\right)=J_1^1J_1^2. \tag{39}$$

First, we have

$$J_1^1=\int_{\Omega_{T^d}}\varphi_1^l(x)dx=T^{\frac{nd}{2}}\int_{|\xi|\le 2}\varphi_1^l(\xi)d\xi=CT^{\frac{nd}{2}}. \tag{40}$$

Also,

$$\begin{aligned}
J_1^2&=\int_0^T\varphi_2^{-\frac{1}{p-1}}(t)\Big|D^{\alpha+2}_{t|T}\varphi_2(t)\Big|^{\frac{p}{p-1}}dt\\
&=T^{1-(\alpha+2)q}\int_0^1(1-s)^{\beta-(\alpha+2)q}ds=CT^{1-(\alpha+2)q}.
\end{aligned} \tag{41}$$

Applying Corollary 1 yields $0<\beta-(\alpha+2)q+\frac{nd}{2}(\alpha=1-\varpi)$, and so the integral J_1^2 exists.

Now, by (39)–(41), we obtain

$$J_1=CT^{1-(\alpha+2)q+\frac{nd}{2}}. \tag{42}$$

For J_2, we have

$$J_2=\left(\int_{\Omega_{T^d}}\varphi_1^l(x)dx\right)\left(\int_0^T\varphi_2^{-\frac{m}{p-m}}(t)\big|b'(t)\big|^{\frac{p}{p-m}}\Big|D^{\alpha}_{t|T}\varphi_2(t)\Big|^{\frac{p}{p-m}}dt\right)=J_2^1J_2^2. \tag{43}$$

It is clear that the estimates for J_2^1 coincide with the above estimate for J_1^1.

Since $(t+1)^{r-1} \sim b'(t), 0 < r < 1$, we obtain

$$\begin{aligned} J_2^2 &= \int_0^1 \left[(1+Ts)^{r-1}\right]^{\frac{p}{p-m}} [1-s]^{-\frac{\beta m}{p-m}} T^{-\alpha\frac{p}{p-m}} (1-s)^{(\beta-\alpha)\frac{p}{p-m}} T ds \\ &= T^{1-\alpha\frac{p}{p-m}} \int_0^1 [1+Ts]^{\frac{(r-1)p}{p-m}} (1-s)^{-\alpha\frac{p}{p-m}+\beta} ds. \end{aligned} \tag{44}$$

We split the integral $\int_0^1 = \int_0^{T^{-k}} + \int_{T^{-k}}^1$ into two integrals, where $k \in (0,1)$. By taking $\beta \gg 1$, we can estimate the first integral as follows

$$\int_0^{T^{-k}} [1+Ts]^{\frac{(r-1)p}{p-m}} (1-s)^{\beta-\alpha\frac{p}{p-m}} ds \leq CT^{-k}. \tag{45}$$

For the second integral, we have

$$[1+Ts]^{\frac{(r-1)p}{p-m}} \leq \left(1+T^{1-k}\right)^{\frac{(r-1)p}{p-m}}.$$

Thus,

$$\begin{aligned} \int_{T^{-k}}^1 [Ts+1]^{\frac{(r-1)p}{p-m}} (1-s)^{-\alpha\frac{p}{p-m}+\beta} ds &\leq \left(T^{1-k}+1\right)^{\frac{(r-1)p}{p-m}} \int_{T^{-k}}^1 (1-s)^{-\alpha\frac{p}{p-m}+\beta} ds \\ &\leq C\left(T^{1-k}+1\right)^{\frac{p(r-1)}{p-m}}. \end{aligned} \tag{46}$$

Set

$$k = \frac{(r-1)p}{p(r-1)-(p-m)}.$$

Then,

$$T^{-k} \sim T^{(1-k)\frac{p(r-1)}{p-m}},$$

and thus

$$J_2 \leq CT^{1-\alpha\frac{p}{p-m} - \frac{p(r-1)}{-(p-m)+p(r-1)} + \frac{nd}{2}}. \tag{47}$$

Now, for J_3, we put

$$J_3 = \left(\int_{\Omega_{T^d}} \varphi_1^l(x) dx\right) \left(\int_0^T \varphi_2^{-\frac{m}{p-m}}(t) |b(t)|^{\frac{p}{p-m}} \left|D_{t|T}^{\alpha+1}\varphi_2(t)\right|^{\frac{p}{p-m}} dt\right) = J_3^1 J_3^2. \tag{48}$$

It clear that the estimates for J_3^1 coincide with the above estimate for J_1^1.

For J_3^2, we know that $b(t) = (t+1)^r$ is a strictly increasing function for $0 < t < 1$ with $0 < r < 1$. Then, for all $0 < t < 1$, we have $b(t) \leq b(T) = (1+T)^r$. Since $p - m > 0$, we obtain

$$b(t)^{\frac{p}{p-m}} \leq (T+1)^{\frac{rp}{p-m}}.$$

So, for a large enough T, we have

$$\begin{aligned} J_3^2 &= \int_0^1 \left[(1-s)_+^{\beta}\right]^{-\frac{m}{p-m}} |b(t)|^{\frac{p}{p-m}} \left[CT^{-\alpha-1}(1-s)^{\beta-\alpha-1}\right]^{\frac{p}{p-m}} ds \\ &\leq (1+T)^{\frac{rp}{p-m}} CT^{1-(\alpha+1)\frac{p}{p-m}} \int_0^1 (1-s)^{\beta-(\alpha+1)\frac{p}{p-m}} ds \\ &\leq CT^{1-(\alpha+1-r)\frac{p}{p-m}}. \end{aligned} \tag{49}$$

Hence,

$$J_3 \leq CT^{1-(\alpha+1-r)\frac{p}{p-m}+\frac{nd}{2}}. \tag{50}$$

Finally, for J_4, we have

$$J_4 = \left(\int_{\Omega_{T^d}} \varphi_1^{l-2q}(x) \left(|\Delta_x \varphi_1(x)|^q + |\nabla_x \varphi_1(x)|^{2q} \right) dx \right) \left(\int_0^T \varphi_2^{-\frac{1}{p-1}}(t) \left| D_{t|T}^{\alpha} \varphi_2(t) \right|^q dt \right) = J_4^1 J_4^2. \tag{51}$$

If we choose l such that $l-2q>0$, we see that φ_1^{l-2q} is bounded, since φ_1 is bounded. We have for $i=1,\ldots,n$

$$\begin{aligned} \partial_{x_i}\varphi_1 = \partial_{x_i}\Phi\left(\tfrac{|x|^2}{\eta}\right) = \tfrac{2}{\eta}\Phi'\left(\tfrac{|x|^2}{\eta}\right)x_i \\ \left|\nabla_x \varphi_1(x)\right|^{2q} = \left(\tfrac{2}{\eta}\right)^{2q} \left|\Phi'\left(\tfrac{|x|^2}{\eta}\right)\right|^{2q} . |x|^{2q}. \end{aligned}$$

Since $\eta = T^{\frac{d}{2}}$ and $|\Phi'(z)| \leq \frac{c}{1+z}$, we obtain

$$\begin{aligned} \int_{R^n} |\nabla_x \varphi_1(x)|^{2q} dx &= \left(\frac{2}{\eta}\right)^{2q} T^{\frac{nd}{2}} \int_{|\xi|^2 \leq 2} \left|\Phi'\left(|\xi|^2\right)\right|^{2q} \left(T^d |\xi|^2\right)^q d\xi \\ &\leq CT^{\frac{nd}{2}-dq} \int_{|\xi|^2\leq 2} \frac{|\xi|^{2q}}{\left(1+|\xi|^2\right)^{2q}} d\xi \leq C_0 T^{\frac{nd}{2}-dq}. \end{aligned} \tag{52}$$

We also have the following relation for all $i=1,\ldots,n$

$$\begin{aligned} \partial_{x_i}^2 \varphi_1 = \partial_{x_i}\left(\tfrac{2}{\eta}\Phi'\left(\tfrac{|x|^2}{\eta}\right)x_i\right) = \tfrac{4}{\eta^2}\Phi''\left(\tfrac{|x|^2}{\eta}\right)x_i^2 + \tfrac{2}{\eta}\Phi'\left(\tfrac{|x|^2}{\eta}\right) \\ \left|\Delta_x \varphi_1(x)\right|^q = \left|\tfrac{4}{\eta^2}\Phi''\left(\tfrac{|x|^2}{\eta}\right)|x|^2 + \tfrac{2n}{\eta}\Phi'\left(\tfrac{|x|^2}{\eta}\right)\right|^q. \end{aligned}$$

Therefore,

$$\int_{R^n} |\Delta_x \varphi_1(x)|^q dx = \int_{R^n} \left| \frac{4}{\eta^2}\Phi''\left(\frac{|x|^2}{\eta}\right)|x|^2 + \frac{2n}{\eta}\Phi'\left(\frac{|x|^2}{\eta}\right) \right|^q dx.$$

Using Minkowski's inequality with a change of variables, we obtain

$$\left(\int\limits_{R^n} |\Delta_x \varphi_1(x)|^q dx\right)^{1/q}$$
$$\leq \left(4^q \eta^{\frac{n}{2}-2q} \int\limits_{2\geq|\xi|^2} \left|\Phi''\left(|\xi|^2\right)\right| \left(\eta|\xi|^2\right)^q d\xi\right)^{1/q} + \left((2n)^q \eta^{\frac{n}{2}-q} \int\limits_{2\geq|\xi|^2 \left|\Phi'\left(|\xi|^2\right)\right| d\xi}\right)^{\frac{1}{q}}$$
$$\leq \left(4^q \eta^{\frac{n}{2}-q} C_1\right)^{\frac{1}{q}} + \left((2n)^q \eta^{\frac{n}{2}-q} C_2\right)^{1/q} \leq C\eta^{\frac{n}{2q}-1},$$

where

$$C_1 = \int\limits_{|\xi|^2\leq 2} \left|\Phi''\left(|\xi|^2\right)\right| \left(\eta|\xi|^2\right)^q d\xi \quad C_2 = \int\limits_{|\xi|^2\leq 2} \left|\Phi'\left(|\xi|^2\right)\right| d\xi.$$

Thus,

$$\int\limits_{R^n} |\Delta_x \varphi_1(x)|^{\frac{p}{p-1}} dx \leq CT^{\frac{nd}{2}-d\frac{p}{p-1}}.$$

Therefore,

$$J_4^1 \leq CT^{\frac{nd}{2}-d\frac{p}{p-1}}. \tag{53}$$

Similarly, we deal with J_4^2.

$$J_4^2 = \int\limits_0^1 (1-s)^{\beta-\alpha\frac{p}{p-1}} ds . T^{-\alpha\frac{p}{p-1}+1} = CT^{-\alpha\frac{p}{p-1}+1}. \tag{54}$$

We conclude that

$$J_4 \leq CT^{-(d+\alpha)\frac{p}{p-1}+\frac{nd}{2}+1}. \tag{55}$$

From (42), (47), (50), and (55) we obtain

$$\begin{aligned}\int\limits_0^T \int\limits_{R^n} |u|^p \Psi dx dt < & C\Big(T^{1-(\alpha+2)q+\frac{nd}{2}} \\ & T^{1-\alpha\frac{p}{p-m}-\frac{(r-1)p}{p(r-1)-(p-m)}+\frac{nd}{2}} \\ & T^{1-(\alpha+1-r)\frac{p}{p-m}+\frac{nd}{2}} \\ & T^{-(\alpha+d)\frac{p}{p-1}+\frac{nd}{2}+1}\Big).\end{aligned} \tag{56}$$

Our next goal is to make the exponents of T negative. It remains to see that

$$-\frac{(\alpha+d)p}{p-1} + \frac{nd}{2} + 1 < 0, \tag{57}$$

and

$$1 - \frac{\alpha p}{p-m} - \frac{(r-1)p}{p(r-1)-(p-m)} + \frac{nd}{2} < 0. \tag{58}$$

By taking $(p-1)q = p, 2\rho = nd$ from (57), we obtain

$$1 + \rho < q(\alpha + \frac{2\rho}{n}), \tag{59}$$

and from (58) with $m < p$, after some calculations, we obtain

$$q > \sqrt{\left(\frac{B}{2A}\right)^2 - \frac{C}{A}} - \frac{B}{2A}, \tag{60}$$

where

$$\begin{aligned} A &= (1-m)(r+m-2)(1+\rho) - \alpha(r+m-2) - (1-r)(1-m), \\ B &= m[(2m-3+r)(1+\rho) + (1-r) + \alpha], \\ C &= -m^2(\rho+1). \end{aligned}$$

Consequently,

$$q > \max\left\{ \frac{1+\rho}{\alpha + \frac{2\rho}{n}}, \sqrt{\left(\frac{B}{2A}\right)^2 - \frac{C}{A}} - \frac{B}{2A} \right\}. \tag{61}$$

Since $\rho = \frac{nd}{2}$ and $q = \frac{p}{p-1}$, we obtain the condition

$$\frac{p}{p-1} > \inf_{d>0} \max\left\{ \frac{\frac{nd}{2}+1}{\alpha + d}, \sqrt{\left(\frac{B}{2A}\right)^2 - \frac{C}{A}} - \frac{B}{2A} \right\}, \tag{62}$$

and from (56) and (62), we find the estimate

$$\int_0^T \int_{R^n} |u|^p \Psi dxdt < CT^{-\chi}, \tag{63}$$

with $\chi = \chi(p, n, m, r, \varpi)$. So, by taking the limit as $T \to \infty$, using the dominated convergence Theorem, and the fact that

$$\lim_{T\to\infty} \Psi = 1 \quad for \quad 0 < t < T,$$

we conclude that

$$\int_0^T \int_{R^n} |u|^p \Psi dxdt = 0.$$

This implies that $u \equiv 0$, which contradicts (26). $\square$

5. Conclusions and Relevance of the Work

One of the main achievements of our research is to show the impact of nonlinear memory on the absence of global solutions even the existence of nonlinear dissipation. By imposing new appropriate conditions and with the help of Fourier transform and fractional derivative calculus, we obtained our unusual results, regarding the local in time existence and blow-up of the solution in finite time.

It is possible to formulate a number of similar problems that are extremely important from the point of view of practical applications and whose solution requires new methods in the literature, namely: problems that contain a fractional derivative in the boundary conditions, with a variable time delay, simplifying the mathematical expression of the condition (62), and reevaluating problem (1) in the case of $r \in (-1, 0]$, see [22–28]. It would be very interesting if one considered numerical studies of this model, which will be our next research project.

Author Contributions: Writing—original draft preparation, S.B.R., A.B. and A.M. (Amer Memou); writing—review and editing, A.H., A.M. (Abdelkader Moumen), H.S. and T.A. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: The researchers would like to thank the Deanship of Scientific Research, Qassim University, for the continous support.

Conflicts of Interest: The authors declare no conflict of interest.

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