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Certain Sharp Coefficient Results on a Subclass of Starlike Functions Defined by the Quotient of Analytic Functions

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Abstract: In the present paper, we consider a subclass of starlike functions $\mathcal{G}_{3/2}$ defined by the ratio of analytic representations of convex and starlike functions. The main aim is to determine the bounds of Fekete–Szegö-type inequalities and Hankel determinants for functions in this class. It is proved that $\max\{|\mathcal{H}_{3,1}(f)| : f \in \mathcal{G}_{3/2}\}$ is equal to $\frac{1}{81}$. The bounds for $f \in \mathcal{G}_{3/2}$ are sharp.

Keywords: Hankel determinant; starlike functions; convex functions; coefficient estimate

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1. Introduction and Definitions

Let $\mathcal{A}$ denote the normalized analytic functions defined in the open unit disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ with the series expansion of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad z \in \mathbb{D}. \tag{1}$$

Let $\mathcal{S} \subseteq \mathcal{A}$ be the class of univalent functions in $\mathbb{D}$. Suppose that $\mathcal{P}$ is the Carathéodory class (see [1]) of all functions p that are analytic in $\mathbb{D}$ with $\Re(p(z)) > 0$ and normalized by

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \quad z \in \mathbb{D}. \tag{2}$$

A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{S}^*$ of starlike functions, if, and only if,

$$\Re \frac{zf'(z)}{f(z)} > 0, \quad z \in \mathbb{D}. \tag{3}$$

For the class $\mathcal{C}$ of convex functions, the necessary and sufficient condition is

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, \quad z \in \mathbb{D}. \tag{4}$$

It is known that if $f \in \mathcal{C}$, then $zf'(z) \in \mathcal{S}^*$—see [2].

Nonlinear functional analysis is an active topic and has its applications in the natural sciences, economics, and numerical analysis—see, for example, [3–7]. In [8], Silverman introduced and studied a new subclass of $\mathcal{A}$ using the quotient of $\left(1 + \frac{zf''(z)}{f'(z)}\right) / \frac{zf'(z)}{f(z)}$. For $b \in (0, 1]$, the class $\mathcal{G}_b$ was defined by

$$\mathcal{G}_b = \left\{ f \in \mathcal{A} : \left| \frac{1 + \frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}} - 1 \right| < b, \quad z \in \mathbb{D} \right\}. \tag{5}$$

In [9], Obradović and Tuneski found that, for $f \in \mathcal{A}$, if

$$\Re\left(\frac{1+\frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}}\right) < \frac{3}{2}, \quad z \in \mathbb{D}, \tag{6}$$

then $f \in \mathcal{S}^*$.

We say g_1 is subordinate to g_2 in $\mathbb{D}$, written as $g_1 \prec g_2$, if $g_1(z) = g_2(\omega(z))$ for some analytic function $\omega(z)$ with $\omega(0) = 0$ and $|\omega(z)| < 1$. If g_2 in $\mathbb{D}$ is univalent, then $g_1 \prec g_2$ is equivalent to

$$g_1(0) = g_2(0) \quad \text{and} \quad g_1(\mathbb{D}) \subset g_2(\mathbb{D}). \tag{7}$$

Using subordination, we can also write

$$\mathcal{G}_b = \left\{ f \in \mathcal{A} : \frac{1+\frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}} \prec 1 + bz, \quad z \in \mathbb{D} \right\}. \tag{8}$$

In [10], the upper bounds on the initial coefficients and Hankel determinants for $f \in \mathcal{G}_b$ were derived. Motivated by the above results, Răducanu [11] investigated the class $\mathcal{G}_{3/2}$ defined by

$$\mathcal{G}_{3/2} := \left\{ f \in \mathcal{A} : \Re\left(\frac{1+\frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}}\right) < \frac{3}{2}, \quad z \in \mathbb{D} \right\}. \tag{9}$$

Clearly, $\mathcal{G}_{3/2}$ is a subclass of starlike functions. With the additional restriction of $f''(0) = p$ for $p \in [0, 2]$, Răducanu obtained the upper bounds of some initial coefficients and the difference of moduli of successive coefficients $|a_3 - a_2|$ and $|a_4 - a_3|$ for $f \in \mathcal{G}_{3/2}$. For other investigations on the analytic functions associated with the ratio of analytic representations of convex and starlike functions, we refer to [12–14].

The Hankel determinant $\mathcal{H}_{q,n}(f)$ with $q, n \in \mathbb{N}$ and $a_1 = 1$ for a function $f \in \mathcal{S}$ of the series form (1) was given by Pommerenke [15,16] as

$$\mathcal{H}_{q,n}(f) := \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}. \tag{10}$$

This is widely used in the theory of Markov processes, of non-stationary signals, in the Hamburger moment problem, and many other issues in both pure mathematics and technical applications—see, for example, [17,18]. In recent years, many results on the upper bounds of Hankel determinant for various subclasses of univalent functions were obtained—interested readers may refer to [19–25] and the references.

In the present paper, we aim to investigate coefficient problems related to Fekete–Szegö-type functionals and Hankel determinants for the functions $f \in \mathcal{G}_{3/2}$.

2. A Set of Lemmas

To prove our main results, we require the following lemmas.

Lemma 1 (see [26])**.** *Let $p \in \mathcal{P}$ be the form of (2). Then,*

$$2c_2 = c_1^2 + x\left(4 - c_1^2\right), \tag{11}$$

$$4c_3 = c_1^3 + 2\left(4 - c_1^2\right)c_1 x - c_1\left(4 - c_1^2\right)x^2 + 2\left(4 - c_1^2\right)\left(1 - |x|^2\right)\delta, \tag{12}$$

$$8c_4 = c_1^4 + \left(4 - c_1^2\right)x\left[c_1^2\left(x^2 - 3x + 3\right) + 4x\right] - 4\left(4 - c_1^2\right)\left(1 - |x|^2\right) \left[c_1(x-1)\delta + \overline{x}\delta^2 - \left(1 - |\delta|^2\right)\rho\right] \tag{13}$$

for some $x, \delta, \rho \in \overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| \leq 1\}$.

Lemma 2 (see [27])**.** *Let $\mu \in \mathbb{C}$. If $p \in \mathcal{P}$ is represented as (2), then*

$$|c_{n+k} - \mu c_n c_k| \leq 2\max(1, |2\mu - 1|). \tag{14}$$

Lemma 3. *For all $(c, t) \in [0, 2) \times \left[\frac{1}{3}, 1\right]$, we have*

$$17(2+c)^2(2-c)t^2 + 2(2+c)\left(23c^2 + 116c - 162\right)t + 29c^3 - 290c^2 + 512 \geq 0. \tag{15}$$

Proof. Let

$$F_1(c,t) := 17(2+c)^2(2-c)t^2 + 2(2+c)\left(23c^2 + 116c - 162\right)t + 29c^3 - 290c^2 + 512. \tag{16}$$

It is noted that

$$F_1(c,t) = (2+c)F_2(c,t) + 29c^3 - 290c^2 + 512, \tag{17}$$

where

$$F_2(c,t) = 17\left(4 - c^2\right)t^2 + 2\left(23c^2 + 116c - 162\right)t. \tag{18}$$

Let $\hat{c}_0 = \frac{\sqrt{7090} - 58}{23} \approx 1.1392$ be the only positive root of the equation $23c^2 + 116c - 162 = 0$. If $c \geq \hat{c}_0$, we have $23c^2 + 116c - 162 \geq 0$ and thus $F_2(c,t) \geq F_2\left(c, \frac{1}{3}\right)$. Then

$$F_1(c,t) \geq (2+c)F_2\left(c, \frac{1}{3}\right) + 29c^3 - 290c^2 + 512 =: s_1(c), \quad c \in [\hat{c}_0, 2). \tag{19}$$

In light of

$$s_1(c) = \frac{2}{9}\left(191c^3 - 836c^2 + 244c + 1400\right) \tag{20}$$

and $s_1'(c) \leq 0$ on $[\hat{c}_0, 2)$, we get that $s_1(c) \geq s_1(2) = 16$. Hence, we obtain that $F_1(c,t) \geq 0$ on $[\hat{c}_0, 2] \times [0, 1]$.

For $c < \hat{c}_0$, we note that the symmetric axis of F_2 is defined by

$$t_0 = \frac{-23c^2 - 116c + 162}{17(4 - c^2)} > 0. \tag{21}$$

Let $\hat{c}_1 = \frac{\sqrt{982} - 29}{3} \approx 0.7790$ be the only positive root of the equation $6c^2 + 116c - 94 = 0$. If $c \leq \hat{c}_1$, we have $t_0 \geq 1$. Then $F_2(c,t) \geq F_2(c,1)$, which leads to

$$F_1(c,t) \geq (2+c)F_1(c,1) + 29c^3 - 290c^2 + 512 = 2c\left(104 + 29c^2\right) \geq 0. \tag{22}$$

If $c \in (\hat{c}_1, \hat{c}_0)$, it is simple to observe that

$$F_1(c,t) \geq 17(2+c)^2(2-c) \cdot \frac{1}{9} + 2(2+c)\left(23c^2 + 116c - 162\right) \cdot 1 + 29c^3 - 290c^2 + 512$$
$$= \frac{2}{9}\left(329c^3 + 136c^2 + 664c - 544\right) > 0$$

on $(\hat{c}_1, \hat{c}_0)$. Hence, $F_1(c,t) \geq 0$ for all $(c,t) \in [0, \hat{c}_0] \times \left[\frac{1}{3}, 1\right]$. Therefore, we deduce that $F_1(c,t) \geq 0$ on $[0,2) \times \left[\frac{1}{3}, 1\right]$. The assertion in Lemma 3 thus follows. □

Lemma 4. *For all* $(c,t) \in [0,2) \times \left[\frac{1}{3}, 1\right]$*, we have*

$$17(1+c)\left(4-c^2\right)t^2 + \left(46c^3 + 162c^2 + 140c - 324\right)t + 29c^3 - 145c^2 + 256 \geq 0. \quad (23)$$

Proof. Define

$$F_3(c,t) := 17(1+c)\left(4-c^2\right)t^2 + \left(46c^3 + 162c^2 + 140c - 324\right)t + 29c^3 - 145c^2 + 256. \quad (24)$$

It is observed that

$$\begin{aligned} F_3(c,t) - F_1(c,t) &= -17\left(4-c^2\right)t^2 + 162\left(2-c^2\right)t + 145c^2 - 256 \\ &\geq -17\left(4-c^2\right)t + 162\left(2-c^2\right)t + 145c^2 - 256 \\ &= (256 - 145c^2)(1-t). \end{aligned}$$

For $c^2 \leq \frac{256}{145}$, it is clear that $F_3(c,t) \geq F_1(c,t) \geq 0$. If $c^2 > \frac{256}{145}$, then $46c^3 + 162c^2 + 140c - 324 > 0$ and it follows that $F_3(c,t) \geq F_3\left(c, \frac{1}{3}\right) > 0$. Hence, we have $F_3(c,t) \geq 0$ on $[0,2) \times \left[\frac{1}{3}, 1\right]$, which leads to the assertion in Lemma 4. □

3. Main Results

For the class of analytic univalent functions, Fekete Szegö studied the maximum value of $|a_3 - \zeta a_2|$ when ζ is real. After that, the upper bound for the functional $|a_3 - \zeta a_2|$ were investigated extensively for various subclasses of univalent functions. For some recent works, see, for example, [28–31]. We begin by finding the Fekete–Szegö-type inequalities for $f \in \mathcal{G}_{3/2}$.

Theorem 1. *Let* $\gamma \in \mathbb{C}$*. If* $f \in \mathcal{G}_{3/2}$ *is of the form* (1)*, then*

$$\left|a_3 - \gamma a_2^2\right| \leq \frac{1}{4}\max\{1, |3 - 4\gamma|\}. \quad (25)$$

The inequality is sharp.

Proof. Let $f \in \mathcal{G}_{3/2}$. Then there exists a function $p \in \mathcal{P}$ such that

$$3\frac{zf'(z)}{f(z)} - 2\left(1 + \frac{zf''(z)}{f'(z)}\right) = p(z)\frac{zf'(z)}{f(z)}, \quad z \in \mathbb{D}. \quad (26)$$

Assuming that

$$p(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + c_4 z^4 + \cdots, \quad z \in \mathbb{D}. \quad (27)$$

Using (1), it follows that

$$\frac{zf'(z)}{f(z)} = 1 + a_2 z + \left(2a_3 - a_2^2\right)z^2 + \left(a_2^3 - 3a_2a_3 + 3a_4\right)z^3$$
$$+ \left(4a_5 - a_2^4 + 4a_2^2a_3 - 4a_2a_4 - 2a_3^2\right)z^4 + \cdots \tag{28}$$

and

$$1 + \frac{zf''(z)}{f'(z)} = 1 + 2a_2z + \left(6a_3 - 4a_2^2\right)z^2 + \left(8a_2^3 - 18a_2a_3 + 12a_4\right)z^3$$
$$+ \left(20a_5 - 16a_2^4 + 48a_2^2a_3 - 32a_2a_4 - 18a_3^2\right)z^4 + \cdots. \tag{29}$$

Combining (26)–(29), we obtain

$$a_2 = -\frac{1}{2}c_1, \tag{30}$$
$$a_3 = -\frac{1}{8}\left(c_2 - 2c_1^2\right), \tag{31}$$
$$a_4 = -\frac{1}{144}\left(8c_3 - 21c_1c_2 + 18c_1^3\right), \tag{32}$$
$$a_5 = -\frac{1}{1152}\left(36c_4 - 27c_2^2 - 80c_1c_3 + 138c_1^2c_2 - 72c_1^4\right). \tag{33}$$

Employing (30) and (31), we have

$$\left|a_3 - \gamma a_2^2\right| = \frac{1}{8}\left|c_2 - 2(1-\gamma)c_1^2\right|. \tag{34}$$

An application of Lemma 2 leads to the assertion in Theorem 1. The equality can be attained by the functions f_1 and f_2 given, respectively, by the equations

$$3\frac{zf'(z)}{f(z)} - 2\left(1 + \frac{zf''(z)}{f'(z)}\right) = \frac{1-z}{1+z}\cdot\frac{zf'(z)}{f(z)}, \quad z \in \mathbb{D}, \tag{35}$$

$$3\frac{zf'(z)}{f(z)} - 2\left(1 + \frac{zf''(z)}{f'(z)}\right) = \frac{1-z^2}{1+z^2}\cdot\frac{zf'(z)}{f(z)}, \quad z \in \mathbb{D}. \tag{36}$$

Using series expansions, f_1 and f_2 are in the form of

$$f_1(z) = z + z^2 + \frac{3}{4}z^3 + \frac{19}{36}z^4 + \frac{101}{288}z^5 + \cdots, \quad z \in \mathbb{D}, \tag{37}$$

$$f_2(z) = z + \frac{1}{4}z^3 + \frac{1}{32}z^5 + \cdots, \quad z \in \mathbb{D}. \tag{38}$$

□

The second Hankel determinant is known as the functional $\mathcal{H}_{2,2}(f) = a_2a_4 - a_3^2$. In recent years, many results on the upper bounds of second Hankel determinant for certain subclass of univalent functions were obtained, see [32–35]. Now, we provide the sharp bound of $\mathcal{H}_{2,2}(f)$ for $f \in \mathcal{G}_{3/2}$.

Theorem 2. *If $f \in \mathcal{G}_{3/2}$, then*

$$|H_{2,2}(f)| = \left|a_2a_4 - a_3^2\right| \leq \frac{1}{16}. \tag{39}$$

The bound is sharp.

Proof. From (30) to (32), we have

$$\mathcal{H}_{2,2}(f) = a_2a_4 - a_3^2 = -\frac{1}{64}c_2^2 - \frac{1}{96}c_1^2c_2 + \frac{1}{36}c_1c_3. \tag{40}$$

Let $f \in \mathcal{G}_{3/2}$ and $f_\theta(z) = e^{-i\theta}f\left(e^{i\theta}z\right)$, $\theta \in \mathbb{R}$. We know $f_\theta(z) \in \mathcal{G}_{3/2}$ for all $\theta \in \mathbb{R}$ and $|\mathcal{H}_{2,2}(f_\theta)| = |\mathcal{H}_{2,2}(f)|$. Since the class $\mathcal{G}_{3/2}$ and the functional $|\mathcal{H}_{2,2}(f)|$ are rotation-invariant, we can assume that a_2 of f is real and, thus, $c_1 := c \in [0,2]$. Using (11) and (12) to express c_2 and c_3, we obtain

$$\begin{aligned}\mathcal{H}_{2,2}(f) = &-\frac{5}{2304}c^4 + \frac{1}{1152}c^2\left(4-c^2\right)x - \frac{1}{2304}\left(36+7c^2\right)\left(4-c^2\right)x^2 \\ &+ \frac{1}{72}c\left(4-c^2\right)\left(1-|x|^2\right)\delta.\end{aligned}$$

Let $|x| := t$. From $|\delta| \le 1$, we get

$$\begin{aligned}|\mathcal{H}_{2,2}(f)| \le &\frac{5}{2304}c^4 + \frac{1}{1152}c^2\left(4-c^2\right)t + \frac{1}{2304}\left(36+7c^2\right)\left(4-c^2\right)t^2 \\ &+\frac{1}{72}c\left(4-c^2\right)\left(1-t^2\right) =: G(c,t).\end{aligned}$$

As $7c^2 - 32c + 36 \ge 0$ on $[0,2]$, it is noted that

$$\frac{\partial G}{\partial t} = \frac{1}{1152}\left(4-c^2\right)\left[\left(7c^2-32c+36\right)t + c^2\right] \ge 0. \tag{41}$$

Thus we have $G(c,t) \le G(c,1)$. Putting $t=1$ leads to

$$|\mathcal{H}_{2,2}(f)| \le \frac{1}{16} - \frac{1}{576}c^4 \le \frac{1}{16}. \tag{42}$$

The bound is achieved by the function f_2 given by Equation (36). □

It is not easy to find the sharp bound of the third Hankel determinant for a certain subclass of univalent functions. For instance, Kowalczyk et al. [36] proved that $|\mathcal{H}_{3,1}(f)| \le \frac{4}{9}$ for $f \in \mathcal{S}^*$. Before it was solved, there are many works investigated this problem—for details, we refer to [37,38]. Now, we give the sharp bound of $\mathcal{H}_{3,1}(f)$ for $f \in \mathcal{G}_{3/2}$.

Theorem 3. *Suppose that $f \in \mathcal{G}_{3/2}$. Then,*

$$|\mathcal{H}_{3,1}(f)| \le \frac{1}{81}. \tag{43}$$

The result is the best possible.

Proof. From the definition, we know

$$\mathcal{H}_{3,1}(f) = 2a_2a_3a_4 - a_4^2 - a_2^2a_5 - a_3^3 + a_3a_5. \tag{44}$$

Let $f \in \mathcal{G}_{3/2}$ and $f_\theta(z) = e^{-i\theta}f\left(e^{i\theta}z\right)$, $\theta \in \mathbb{R}$. It is observed that $f_\theta(z) \in \mathcal{G}_{3/2}$ for all $\theta \in \mathbb{R}$ and $|\mathcal{H}_{3,1}(f_\theta)| = |\mathcal{H}_{3,1}(f)|$. As the class $\mathcal{G}_{3/2}$ and the functional $|\mathcal{H}_{3,1}(f)|$ are invariant under the rotations, we may assume that a_2 of f is real and, thus, $c_1 := c \in [0,2]$. Substituting (30)–(33) into (44), we have

$$\mathcal{H}_{3,1}(f) = \frac{1}{82944}\left(-81c_2^3 + 48cc_2c_3 + 324c_2c_4 - 256c_3^2 + 18c^2c_2^2\right). \tag{45}$$

Let $b = 4 - c^2$. By applying Lemma 1 and inserting the formulae of c_2, c_3 and c_4 into (45), we obtain

$$\begin{aligned}\mathcal{H}_{3,1}(f) = \frac{1}{663552}\Big\{&37c^6 - 81x^3b^3 + 648x^3b^2 + 648c^2x^2b - 278c^4x^2b \\ &+ 162c^4x^3b - 137c^2x^2b^2 - 22c^2x^3b^2 + 34c^2x^4b^2 + 109c^4xb \\ &+ 232c^3b\left(1-|x|^2\right)\delta - 648|x|^2b^2\left(1-|x|^2\right)\delta^2 - 512b^2\left(1-|x|^2\right)^2\delta^2 \\ &- 136cx^2b^2(1-|x|^2)\delta - 648c^2\overline{x}b(1-|x|^2)\delta^2 - 280cxb^2\left(1-|x|^2\right)\delta \\ &- 648c^3xb(1-|x|^2)\delta + 648c^2b\left(1-|x|^2\right)\left(1-|\delta|^2\right)\rho \\ &+648xb^2\left(1-|x|^2\right)\left(1-|\delta|^2\right)\rho\Big\},\end{aligned}$$

where $x, \delta, \rho \in \overline{\mathbb{D}}$. By suitable rearrangements, it is observed that $\mathcal{H}_{3,1}(f)$ can be written as

$$\mathcal{H}_{3,1}(f) = \frac{1}{663552}\Big[u_1(c,x) + u_2(c,x)\delta + u_3(c,x)\delta^2 + \Phi(c,x,\delta)\rho\Big], \tag{46}$$

where

$$\begin{aligned}u_1(c,x) &= 37c^6 + \left(4-c^2\right)x\Big[34c^2\left(4-c^2\right)x^3 + \left(103c^4 - 88c^2 + 1296\right)x^2 \\ &\quad +c^2\left(100 - 141c^2\right)x + 109c^4\Big], \\ u_2(c,x) &= 8(4-c^2)\left(1-|x|^2\right)c\Big[-17\left(4-c^2\right)x^2 - \left(46c^2+140\right)x + 29c^2\Big], \\ u_3(c,x) &= 8(4-c^2)\left(1-|x|^2\right)\Big[\left(4-c^2\right)\left(-17|x|^2 - 64\right) - 81c^2\overline{x}\Big], \\ \Phi(c,x,\delta) &= 648(4-c^2)\left(1-|x|^2\right)\left(1-|\delta|^2\right)\Big[\left(4-c^2\right)x + c^2\Big].\end{aligned}$$

Let $|x| =: t$ and $|\delta| =: y$. By observing $|\rho| \le 1$, it follows that

$$\begin{aligned}|\mathcal{H}_{3,1}(f)| &\le \frac{1}{663552}\Big[|u_1(c,x)| + |u_2(c,x)|y + |u_3(c,x)|y^2 + |\Phi(c,x,\delta)|\Big] \\ &\le \frac{1}{663552}[\Gamma(c,t,y)],\end{aligned} \tag{47}$$

where

$$\Gamma(c,t,y) = \sigma_1(c,t) + \sigma_2(c,t)y + \sigma_3(c,t)y^2 + \sigma_4(c,t)\left(1-y^2\right), \tag{48}$$

with

$$\begin{aligned}\sigma_1(c,t) &= 37c^6 + (4-c^2)\Big[34c^2\left(4-c^2\right)t^4 + \left(103c^4 - 88c^2 + 1296\right)t^3 \\ &\quad +c^2\left(100 + 141c^2\right)t^2 + 109c^4t\Big], \\ \sigma_2(c,t) &= 8(4-c^2)\left(1-t^2\right)c\Big[17\left(4-c^2\right)t^2 + \left(46c^2+140\right)t + 29c^2\Big], \\ \sigma_3(c,t) &= 8(4-c^2)\left(1-t^2\right)\Big[\left(4-c^2\right)\left(17t^2+64\right) + 81c^2t\Big], \\ \sigma_4(c,t) &= 648(4-c^2)\left(1-t^2\right)\Big[\left(4-c^2\right)t + c^2\Big].\end{aligned}$$

The inequality (47) comes from the fact that $|u_j(c,x)| \le \sigma_j(c,t)$ for $j = 1,2,3$ and $|\Phi(c,x,\delta)| \le \sigma_4(c,t)\left(1-|\delta|^2\right)$. Here, $|u_1(c,x)| \le \sigma_1(c,t)$ follows from $4 - c^2 \ge 0$ and $103c^4 - 88c^2 + 1296 \ge 0$ on $[0,2]$.

Now the problem reduces to find the maximum value of Γ in the closed cuboid $\Omega := [0,2] \times [0,1] \times [0,1]$. By noting that $\Gamma(0,0,1) = 8192$, we have $\max_{(c,t,y)\in\Omega}\Gamma(c,t,y) \geq 8192$. In the following, we show that $\max_{(c,t,y)\in\Omega}\Gamma(c,t,y) = 8192$.

On the face, $c = 2$, $\Gamma(2,t,y) \equiv 2368$. For $t = 1$, we have

$$\Gamma(c,1,y) = -282c^6 + 1128c^4 - 704c^2 + 5184 =: r_1(c). \tag{49}$$

Since $r_1'(c) = 0$ has two positive roots $\widetilde{c}_1 = \sqrt{\frac{4}{3} - \frac{20\sqrt{141}}{141}} \approx 0.6007$ and $\widetilde{c}_2 = \sqrt{\frac{4}{3} + \frac{20\sqrt{141}}{141}} \approx 1.5185$, r_1 attains its maximum value about 6100.85 on $c = \widetilde{c}_2$ for $c \in [0,2]$. Thus, we may only consider the cases of $c < 2$ and $t < 1$.

First we suppose that $(c,t,y) \in [0,2) \times [0,1) \times (0,1)$. By differentiating partially Γ with respect to y, we obtain

$$\frac{\partial\Gamma}{\partial y} = \sigma_2(c,t) + 2[\sigma_3(c,t) - \sigma_4(c,t)]y. \tag{50}$$

Taking $\frac{\partial\Gamma}{\partial y} = 0$, we obtain

$$y = \frac{17c\left(4-c^2\right)t^2 + \left(46c^3 + 140c\right)t + 29c^3}{2(1-t)[17(4-c^2)t + 145c^2 - 256]} =: y_0. \tag{51}$$

For $y_0 \in (0,1)$, it is possible only if

$$17(2+c)^2(2-c)t^2 + 2(2+c)\left(23c^2 + 116c - 162\right)t + 29c^3 - 290c^2 + 512 < 0 \tag{52}$$

and

$$c^2 > \frac{4(64-17t)}{145-17t}. \tag{53}$$

Now we have to obtain the solutions that satisfy both inequalities (52) and (53) for the existence of critical points with $y \in (0,1)$. From Lemma 3, we know (52) is impossible to hold for $t \in \left[\frac{1}{3}, 1\right]$.

Let $\left(\tilde{c}, \tilde{t}, \tilde{y}\right)$ be a critical point of Γ lies in $[0,2) \times [0,1) \times (0,1)$. From the above discussions, we know $\tilde{t} < \frac{1}{3}$. Assume that $g(t) = \frac{4(64-17t)}{145-17t}$. As $g'(t) < 0$ in $[0,1]$, $g(t)$ is decreasing over $[0,1]$. Hence, $\tilde{c}^2 \geq \frac{350}{209}$. Under the conditions that $t \in \left[0, \frac{1}{3}\right)$, $c \in \left(\sqrt{\frac{350}{209}}, 2\right)$ and $y \in (0,1)$, we find that

$$\sigma_1(c,t) \leq \sigma_1\left(c, \frac{1}{3}\right) =: \tau_1(c) \tag{54}$$

and

$$\sigma_j(c,x) \leq \frac{9}{8}\sigma_j\left(c, \frac{1}{3}\right) =: \tau_j(c), \quad j = 2,3,4. \tag{55}$$

Therefore, we have

$$\Gamma(c,t,y) \leq \tau_1(c) + \tau_2(c)y + \tau_3(c)y^2 + \tau_4(c)\left(1-y^2\right) =: \Xi(c,y). \tag{56}$$

Since $\frac{\partial\Xi}{\partial y} = \tau_2(c) + 2[\tau_3(c) - \tau_4(c)]y$, we have

$$\frac{\partial\Xi}{\partial y}|_{y=1} = \tau_2(c) + 2[\tau_3(c) - \tau_4(c)] = \frac{16}{9}(4-c^2)r_2(c),$$

where $r_2(c) = 191c^3 - 836c^2 + 244c + 1400$. It is easy to get that $r_2(c) \geq 0$ for $c \in [0,2]$. Thus, we have $\frac{\partial \Xi}{\partial y}|_{y=1} \geq 0$. As $\frac{\partial \Xi}{\partial y}|_{y=0} = \tau_2(c) \geq 0$ and $\frac{\partial \Xi}{\partial y}$ is a continuous linear function with respect to y, we deduce that

$$\frac{\partial \Xi}{\partial y} \geq \min\left\{\frac{\partial \Xi}{\partial y}|_{y=0}, \frac{\partial \Xi}{\partial y}|_{y=1}\right\} \geq 0, \quad y \in (0,1). \tag{57}$$

Thus,

$$\Xi(c,y) \leq \Xi(c,1) = \tau_1(c) + \tau_2(c) + \tau_3(c) =: r_3(c). \tag{58}$$

By numerical calculation, we know r_3 has a maximum value about 7391.58 on $c = \sqrt{\frac{350}{209}} \approx 1.2941$ for $c \in \left(\sqrt{\frac{350}{209}}, 2\right)$. Therefore, we have $\Gamma(c,t,y) < 8192$ on $(c,t,y) \in \left(\sqrt{\frac{350}{209}}, 2\right) \times \left[0, \frac{1}{3}\right) \times (0,1)$. Hence, $\Gamma(\tilde{c}, \tilde{t}, \tilde{y}) < 8192$. Thus, any critical point which lies in Ω with $y \in (0,1)$, has a maximum value less than $\Gamma(0,0,1)$. Then it is sufficient to consider the face $y = 0$ and $y = 1$ of Ω to get the maximum value of Γ.

Since

$$\Gamma(c,t,0) = \sigma_1(c,t) + \sigma_4(c,t) \tag{59}$$

and

$$\Gamma(c,t,1) = \sigma_1(c,t) + \sigma_2(c,t) + \sigma_3(c,t), \tag{60}$$

we note that

$$\begin{aligned}\Gamma(c,t,1) - \Gamma(c,t,0) &= \sigma_2(c,t) + \sigma_3(c,t) - \sigma_4(c,t) \\ &=: 8\left(4 - c^2\right)\left(1 - t^2\right)\Theta(c,t),\end{aligned}$$

where

$$\Theta(c,t) = 17(1+c)\left(4 - c^2\right)t^2 + \left(46c^3 + 162c^2 + 140c - 324\right)t + 29c^3 - 145c^2 + 256. \tag{61}$$

From Lemma 4, we know $\Theta(c,t) \geq 0$ on $[0,2) \times \left[\frac{1}{3}, 1\right]$; thus, we have

$$\Gamma(c,t,0) \leq \Gamma(c,t,1), \quad (c,t) \in [0,2) \times \left[\frac{1}{3}, 1\right). \tag{62}$$

If $t < \frac{1}{3}$, from (54) and (55), we obtain

$$\Gamma(c,t,0) \leq \tau_1(c) + \tau_4(c) =: r_4(c). \tag{63}$$

Since

$$r_4(c) = \frac{2}{81}\left(-745c^6 - 8908c^4 + 34592c^2 + 147744\right) \tag{64}$$

has a maximum value about 4376.24 attained on $c \approx 1.2707$ and $\Gamma(0,0,1) = 8192$, we have

$$\Gamma(c,t,0) < \Gamma(0,0,1), \quad (c,t) \in [0,2) \times \left[0, \frac{1}{3}\right). \tag{65}$$

Combining (62) and (65), it remains to find the maximum value of Γ on the face of $y = 1$.

On the face $y = 1$, it is noted that

$$\begin{aligned}\Gamma(c,t,1) &= 37c^6 + \left(4 - c^2\right)\left(\lambda_4 t^4 + \lambda_3 t^3 + \lambda_2 t^2 + \lambda_1 t + \lambda_0\right) \\ &=: Q(c,t),\end{aligned}$$

where

$$\lambda_4 = 34(4-c^2)\left(c^2-4c-4\right),$$
$$\lambda_3 = 103c^4 - 368c^3 - 736c^2 - 1120c + 1296,$$
$$\lambda_2 = 141c^4 - 368c^3 + 476c^2 + 544c - 1504,$$
$$\lambda_1 = 109c^4 + 368c^3 + 648c^2 + 1120c,$$
$$\lambda_0 = 232c^3 - 512c^2 + 2048.$$

In virtue of $c^2 - 4c - 4 \leq 0$ on $[0,2]$, we have $\lambda_4 \leq 0$ and thus

$$Q(c,t) \leq 37c^6 + \left(4-c^2\right)\left(\lambda_3 t^3 + \lambda_2 t^2 + \lambda_1 t + \lambda_0\right) =: W(c,t). \tag{66}$$

Now we need to find the maximum value of W on $[0,2]\times[0,1]$. On the vertices $(0,0)$, $(2,0)$, $(0,1)$ and $(2,1)$, we have $W(0,0) = 8192$, $W(2,0) = W(2,1) = 2368$ and $W(0,1) = 7360$. On the sides of $[0,2]\times[0,1]$, if $c = 0$, we have

$$W(0,t) = 5184t^3 - 6016t^2 + 8192 =: r_5(t). \tag{67}$$

Since $r_5'(t) = 0$ has only one positive root $\widetilde{t}_1 = \frac{188}{243} \approx 0.7737$, we see the maximum value of r_5 is 8192 attained on $t = 0$. If $c = 2$, then $W(2,t) \equiv 2368$ for all $t \in [0,1]$. It is left to discuss the case of $(c,t) \in (0,2)\times(0,1)$. Solving the system of equations

$$\frac{\partial W}{\partial t} = (4-c^2)\left(3\lambda_3 t^2 + 2\lambda_2 t + \lambda_1\right) = 0 \tag{68}$$

and

$$\frac{\partial W}{\partial c} = \mu_3(c)t^3 + \mu_2(c)t^2 + \mu_1(c)t + \mu_0 = 0, \tag{69}$$

where

$$\mu_3 = -618c^5 + 1840c^4 + 4592c^3 - 1056c^2 - 8480c - 4480,$$
$$\mu_2 = -846c^5 + 1840c^4 + 352c^3 - 6048c^2 + 6816c + 2176,$$
$$\mu_1 = -654c^5 - 1840c^4 - 848c^3 + 1056c^2 + 5184c + 4480,$$
$$\mu_0 = 2c\left(111c^4 - 580c^3 + 1024c^2 + 1392c - 4096\right),$$

it is found that all the three critical points lie in $(0,2)\times(0,1)$ are $\left(\bar{c}_1,\bar{t}_1\right) \approx (0.4480, 0.4137)$, $\left(\bar{c}_2,\bar{t}_2\right) \approx (1.0708, 0.7129)$ and $\left(\bar{c}_3,\bar{t}_3\right) \approx (0.4973, 0.9759)$. As $W\left(\bar{c}_1,\bar{t}_1\right) \approx 7913.01$, $W\left(\bar{c}_2,\bar{t}_2\right) \approx 8167.95$ and $W\left(\bar{c}_3,\bar{t}_3\right) \approx 7823.08$, we see the maximum value of W is less than 8192 in $(0,2)\times(0,1)$. In conclusion, we obtain that $W(c,t) \leq 8192$ on $[0,2]\times[0,1]$, which leads to $\Gamma(c,t,y) \leq 8192$ for all $(c,t,y) \in [0,2]\times[0,1]\times[0,1]$. Hence, we obtain

$$|\mathcal{H}_{3,1}(f)| \leq \frac{8192}{663552} = \frac{1}{81}. \tag{70}$$

For the sharpness, we consider the function f_3 satisfying the equation

$$3\frac{zf'(z)}{f(z)} - 2\left(1 + \frac{zf''(z)}{f'(z)}\right) = \frac{1-z^3}{1+z^3}\cdot\frac{zf'(z)}{f(z)}, \quad z \in \mathbb{D}. \tag{71}$$

The series expansion of f_3 is given by

$$f_3(z) = z - \frac{1}{2}z^2 + \frac{1}{4}z^3 - \frac{1}{72}z^4 - \frac{11}{144}z^5 + \cdots, \quad z \in \mathbb{D}. \tag{72}$$

This completes the proof Theorem 3. □

4. Conclusions

In this paper, we investigated some coefficient-related problems for the class $\mathcal{G}_{3/2}$, which is defined by the ratio of analytic representations of convex and starlike functions. Fekete–Szegö-type inequalities and upper bounds for the second and third Hankel determinant were obtained. The calculation is based on the relationship between the coefficients of functions in the considered class and the coefficients for the corresponding Carathéodory class. The results may inspire further investigations on getting more sharp bounds related with functional such as $a_3a_5 - a_4^2$, $a_4 - a_2a_3$ and $a_5 - a_3^2$. By making use of the concept of q-calculus, it is also interesting to investigate a more general class of this type.

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References

1. Carathéodory, C. Über den Variabilitatsbereich der Koeffizienten von Potenzreihen, die gegebene werte nicht annehmen. *Math. Ann.* **1907**, *64*, 95–115. [CrossRef]
2. Goodman, A.W. *Univalent Functions*; Mariner: Tampa, FL, USA, 1983.
3. Ji, E.K.; Kwang, H.S. A regularity of split-biquaternionic-valued functions in Clifford analysis. *J. Nonlinear Sci. Appl.* **2016**, *9*, 6302–6311.
4. Ji, E.K.; Kwang, H.S. Hyper-conjugate harmonic function of conic regular functions in conic quaternions. *East Asian Math. J.* **2015**, *31*, 127–134.
5. Liu, Q.; Dai, B.; Katib, I.; Alhamami, M.A. Financial accounting measurement model based on numerical analysis of rigid normal differential equation and rigid generalised functional equation. *Appl. Math. Nonlinear Sci.* **2022**, *7*, 541–548. [CrossRef]
6. Sun, R.; Sharaf, S.; Ali, B. Human gait modelling and tracking based on motion functionalisation. *Appl. Math. Nonlinear Sci.* **2022**, *7*, 21–30. [CrossRef]
7. An, X.; Yang, R.; Alghazzawi, D.; Joseph, N. Mathematical function data model analysis and synthesis system based on short-term human movement. *Appl. Math. Nonlinear Sci.* **2022**, *7*, 49–58. [CrossRef]
8. Silverman, H. Convex and starlike criteria. *Int. J. Math. Math. Sci.* **1999**, *22*, 75–79. [CrossRef]
9. Obradović, M.; Tuneski, N. On the starlike criteria defined by Silverman. *Zesz. Nauk. Politech. Rzesz. Mat.* **2000**, *24*, 59–64.
10. Park, J.H.; Kumar, V.; Cho, N.E. Sharp coefficient bounds for the quotient of analytic functions. *Kyungpook Math. J.* **2018**, *58*, 231–242. [CrossRef]
11. Răducanu, D. Coefficient estimates for a subclass of starlike functions. *Mathematics* **2020**, *8*, 1646. [CrossRef]
12. Tuneski, N. On the quotient of the representations of convexity and starlikeness. *Math. Nachrichten* **2003**, *248*, 200–203. [CrossRef]
13. Tuneski, N. On a criteria for starlikeness of analytic functions. *Integral Transform. Spec. Funct.* **2003**, *14*, 263–270. [CrossRef]
14. Singh, V.; Nikola, T. On criteria for starlikeness and convexity of analytic functions. *Acta Math. Sci.* **2004**, *24*, 597–602. [CrossRef]
15. Pommerenke, C. On the coefficients and Hankel determinants of univalent functions. *J. Lond. Math. Soc.* **1966**, *1*, 111–122. [CrossRef]
16. Pommerenke, C. On the Hankel determinants of univalent functions. *Mathematika* **1967**, *14*, 108–112. [CrossRef]
17. Dienes, P. *The Taylor Series: An Introduction to the Theory of Functions of a Complex Variable*; Dover: New York, NY, USA, 1957.
18. Srivastava, H.M.; Kaur, G.; Singh, G. Estimates of the fourth Hankel determinant for a class of analytic functions with bounded turnings involving cardioid domains. *J. Nonlinear Convex Anal.* **2021**, *22*, 511–526.
19. Arif, M.; Barukab, O.M.; Khan, S.A.; Abbas, M. The sharp bounds of Hankel determinants for the families of three-leaf-type analytic functions. *Fractal Fract.* **2022**, *6*, 291. [CrossRef]
20. Kowalczyk, B.; Lecko, A.; Sim, Y.J. The sharp bound of the Hankel determinant of the third kind for convex functions. *Bull. Aust. Math. Soc.* **2018**, *97*, 435–445. [CrossRef]
21. Lecko, A.; Sim, Y.J.; Śmiarowska, B. The sharp bound of the Hankel determinant of the third kind for starlike functions of order 1/2. *Complex Anal. Oper. Theory* **2019**, *13*, 2231–2238. [CrossRef]
22. Shi, L.; Shutaywi, M.; Alreshidi, N.; Arif, M.; Ghufran, M.S. The sharp bounds of the third-order Hankel determinant for certain analytic functions associated with an eight-shaped domain. *Fractal Fract.* **2022**, *6*, 223. [CrossRef]

23. Shi, L.; Srivastava, H.M.; Rafiq, A.; Arif, M.; Ihsan, M. Results on Hankel determinants for the inverse of certain analytic functions subordinated to the exponential function. *Mathematics* **2022**, *10*, 3429. [CrossRef]
24. Al-shbeil, I.; Gong, J.; Khan, S.; Khan, N.; Khan, A.; Khan, M.F.; Goswami, A. Hankel and symmetric Toeplitz determinants for a new subclass of q-starlike functions. *Fractal Fract.* **2022**, *6*, 658. [CrossRef]
25. Wang, Z.-G.; Raza, M.; Arif, M.; Ahmad, K. On the third and fourth Hankel determinants of a subclass of analytic functions. *Bull. Malays. Math. Sci. Soc.* **2022**, *45*, 323–359. [CrossRef]
26. Kwon, O.S.; Lecko, A.; Sim, Y.J. On the fourth coefficient of functions in the Carathéodory class. *Comput. Methods Funct. Theory* **2018**, *18*, 307–314. [CrossRef]
27. Libera, R.J.; Złotkiewicz, E.J. Coefficient bounds for the inverse of a function with derivative in $\mathcal{P}$, II. *Proc. Am. Math. Soc.* **1984**, *92*, 58–60.
28. Seoudy, T.; Aouf, M.K. Fekete-Szegö problem for certain subclass of analytic functions with complex order defined by q-analogue of Ruscheweyh Operator. *Constr. Math. Anal.* **2020**, *3*, 36–44. [CrossRef]
29. Baksa, V.; Bandura, A.; Skaskiv, O. Growth estimates for analytic vector-valued functions in the unit ball having bounded L-index in joint variables. *Constr. Math. Anal.* **2020**, *3*, 9–19.
30. Aouf, M.K.; Seoudy, T. Certain class of bi-Bazilevic functions with bounded boundary rotation involving Salăgeăn Operator, *Constr. Math. Anal.* **2020**, *3*, 139–149.
31. Srivastava, H.M.; Wanas, A.K.; Güney, H.Ö. New families of bi-univalent functions associated with the Bazilevič functions and the λ-pseudo-starlike functions. *Iran. J. Sci. Technol. Trans. A Sci.* **2021**, *45*, 1799–1804. [CrossRef]
32. Kowalczyk, B.; Lecko, A. Second Hankel determinant of logarithmic coefficients of convex and starlike functions. *Bull. Aust. Math. Soc.* **2022**, *105*, 458–467. [CrossRef]
33. Sunthrayuth, P.; Aldawish, I.; Arif, M.; Abbas, M.; El-Deeb, S. Estimation of the second-order Hankel determinant of logarithmic coefficients for two subclasses of starlike functions. *Symmetry* **2022**, *14*, 2039. [CrossRef]
34. Sümer Eker, S.; Şeker, B.; Çekiç, B.; Acu, M. Sharp bounds for the second Hankel determinant of logarithmic coefficients for strongly starlike and strongly convex functions. *Axioms* **2022**, *11*, 369. [CrossRef]
35. Sim, Y.J.; Lecko, A.; Thomas, D.K. The second Hankel determinant for strongly convex and Ozaki close-to-convex functions. *Ann. Mat. Pura Appl. (1923-)* **2021**, *200*, 2515–2533. [CrossRef]
36. Kowalczyk, B.; Lecko, A.; Thomas, D.K. The sharp bound of the third Hankel determinant for starlike functions. *Forum Math.* **2022**, *34*, 1249–1254. [CrossRef]
37. Kwon, O.S.; Lecko, A.; Sim, Y.J. The bound of the Hankel determinant of the third kind for starlike functions. *Bull. Malays. Math. Sci. Soc.* **2019**, *42*, 767–780. [CrossRef]
38. Zaprawa, P.; Obradović, M.; Tuneski, N. Third Hankel determinant for univalent starlike functions. *Rev. Real Acad. Cienc. Exactas Fis. Nat. Ser. A Mat.* **2021**, *115*, 1–6. [CrossRef]