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Lyapunov-Type Inequalities for Systems of Riemann-Liouville Fractional Differential Equations with Multi-Point Coupled Boundary Conditions

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Abstract: We consider a system of Riemann–Liouville fractional differential equations with multi-point coupled boundary conditions. Using some techniques from matrix analysis and the properties of the integral operator defined on two Banach spaces, we establish some Lyapunov-type inequalities for the problem considered. Moreover, the comparison between two Lyapunov-type inequalities is given under certain special conditions. The inequalities obtained compliment the existing results in the literature.

Keywords: fractional derivative and integral operators; Green's function; spectral radius; coupled system

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1. Introduction

We are interested in the systems of nonlinear fractional differential equations with multi-point coupled boundary conditions

$$\begin{cases} D_{a+}^{\beta_1} u(t) + f(t, u(t), v(t)) = 0, \ \ t \in (a, b), \\ D_{a+}^{\beta_2} v(t) + g(t, u(t), v(t)) = 0, \ \ t \in (a, b), \\ u(a) = 0, \ u(b) = \sum_{i=1}^{n} a_{1i} u(\xi_i) + \sum_{j=1}^{n} a_{2j} v(\eta_j), \\ v(a) = 0, \ v(b) = \sum_{i=1}^{n} a_{3i} u(\xi_i) + \sum_{j=1}^{n} a_{4j} v(\eta_j), \end{cases} \tag{1}$$

where $a, b \in \mathbb{R}$, $0 < a < b$, $a < \xi_1 < \xi_2 < \cdots < \xi_n < b$, $a < \eta_1 < \eta_2 < \cdots < \eta_n < b$, $a_{ij} \geq 0 (i = 1, 2, 3, 4; j = 1, 2, \ldots, n)$, $1 < \beta_i \leq 2 (i = 1, 2)$, $D_{a+}^{\beta_i} (i = 1, 2)$ is the Riemann–Liouville fractional derivative, and $f, g : [a, b] \times \mathbb{R}^2 \to \mathbb{R}$ are given functions. Using the spectral radius of the matrix, we establish Lyapunov-type inequality for (1). The well-known Lyapunov inequality [1] shows that a necessary condition for the second-order linear differential equation

$$\begin{cases} y''(t) + q(t) y(t) = 0, \ \ a < t < b, \\ u(a) = u(b) = 0 \end{cases}$$

to have nontrivial solutions is that

$$\int_a^b |q(s)| ds > \frac{4}{b - a}. \tag{2}$$

Lyapunov inequality has found many practical applications such as estimates for intervals of disconjugacy [2] and eigenvalue problems [3] in investigating the qualitative proper-

ties of solutions of differential equations, marking the difference between Equations (4) and integral Equations (5). Since then, inequality (2) was rediscovered and generalized many times; see, for example [2–5].

The first Lyapunov-type inequalities for fractional boundary value problems is due to Ferreira [6], where he established the following result:

If a nontrivial solution to the Riemann–Liouville fractional boundary value problem

$$\begin{cases} D^{\alpha}_{a^+} y(t) + q(t)y(t) = 0, \ \ a < t < b, \ \ 1 < \alpha < 2, \\ u(a) = u(b) = 0 \end{cases} \tag{3}$$

exists, where q is a real and continuous function, then

$$\int_a^b |q(s)|ds > \Gamma(\alpha)\left(\frac{4}{b-a}\right)^{\alpha-1}.$$

By substituting the Riemann–Liouville fractional derivative $D^{\alpha}_{a^+}$ in (3) by the Caputo fractional derivative ${}^{C}D^{\alpha}_{a^+}$, a Lyapunov-type inequality [7] was obtained as follows:

If $q \in C[a,b]$, then

$$\int_a^b |q(s)|ds > \frac{\Gamma(\alpha)\alpha^{\alpha}}{[(\alpha-1)(b-a)]^{\alpha-1}}$$

holds if there is a nontrivial solution for the following Caputo boundary value problem

$$\begin{cases} {}^{C}D^{\alpha}_{a^+} y(t) + q(t)y(t) = 0, \ \ a < t < b, \ \ 1 < \alpha < 2, \\ u(a) = u(b) = 0. \end{cases}$$

In [8], Ferreira addresses the issue of further research directions for Lyapunov-type inequality. After that, some results related to the study of Lyapunov-type inequalities for various types of fractional differential equations were obtained; see the survey article of Ntouyas et al. [9] and its complemented survey [10,11], and the papers of [12–15] and references therein. For example, in [12], a Lyapunov-type inequality was obtained for a higher order Riemann–Liouville fractional differential equation with fractional integral boundary conditions, and the lower bound for the eigenvalues of nonlocal boundary value problems was also presented. In [13], the authors proved a Lyapunov-type inequality for a class of Riemann–Liouville fractional boundary value problems with fractional boundary conditions. In [14], a Lyapunov-type inequality was obtained for Riemann–Liouville-type fractional boundary value problems with fractional boundary conditions

$$\begin{cases} D^{\alpha}_{a+} y(t) + q(t)y(t) = 0, \quad t \in (a,b), \\ y(a) = D^{\beta}_{a+} u(b) = 0, \end{cases} \tag{4}$$

where $1 < \alpha \leq 2$, $0 \leq \beta \leq 1$, $q \in C[a,b]$. It was proved that if (4) has a nontrivial solution, then

$$\int_a^b (b-s)^{\alpha-\beta-1}|q(s)|ds > \frac{\Gamma(\alpha)}{(b-a)^{\beta-1}}, \quad \text{if } \ 0 < \alpha - \beta < 1,$$

and

$$\int_a^b |q(s)|ds > \frac{\Gamma(\alpha)2^{2\alpha-\beta-2}}{(b-a)^{\alpha-1}}, \quad \text{if } \ 1 \leq \alpha - \beta < 2.$$

In [15], Wang et al. considered multi-point boundary value problem of the form

$$\begin{cases} D^{\alpha}_{a+} u(t) + q(t)u(t) = 0, \quad t \in (a,b), \ \ 2 < \alpha < 3, \\ u(a) = u'(a) = 0, \ \ D^{\beta+1}_{a+} u(b) = \sum\limits_{i=1}^{m-2} b_i D^{\beta}_{a+} u(\xi_i), \end{cases}$$

where D^{α}_{a+} is the Riemann–Liouville fractional derivative, $a < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < b$, $b_i \geq 0$, $0 \leq \delta = \sum_{i=1}^{m-2} b_i(\xi_i - a)^{\alpha-\beta-1} < (\alpha-\beta-2)(b-a)^{\alpha-\beta-2}$, and obtained the following Lyapunov-type inequalities:

$$\int_a^b (b-s)^{\alpha-\beta-2}\left[(b-a)^{\beta+1} - (b-s)^{\beta+1} + \frac{(b-a)^{\alpha-1}}{\delta}\sum_{i=1}^{m-2} b_i(s-a)\right]|q(s)|ds \geq \Gamma(\alpha).$$

Recently, Jleli, O'Regan and Same [16] studied a coupled system of Caputo fractional differential equations

$$\begin{cases} -({}^{C}D^{\alpha}_{a+}u)(t) = f(t,u,v), \ \ t \in (a,b), \\ -({}^{C}D^{\beta}_{a+}v)(t) = g(t,u,v), \ \ t \in (a,b), \\ u(a) = u(b) = 0, \ \ v(a) = v(b) = 0, \end{cases} \tag{5}$$

where $1 < \alpha, \beta < 2$. Let $I^b_a(\theta, h) = \dfrac{(\theta-1)^{\theta-1}(b-a)^{\theta-1}}{\theta^{\theta}\Gamma(\theta)}\displaystyle\int_a^b h(s)ds$ for $h \in C[a,b]$. They proved the following Lyapunov-type inequalities:

$$I^b_a(\alpha, p_{11}) + I^b_a(\beta, p_{22}) + \left[\left(I^b_a(\alpha, p_{11}) - I^b_a(\beta, p_{22})\right)^2 + 4I^b_a(\alpha, p_{12})I^b_a(\beta, p_{21})\right]^{\frac{1}{2}} \geq 2,$$

if (5) has a nontrivial solution and there exist positive functions $p_{ij} \in C[a,b]$ $(i,j = 1,2)$ such that

$$|f(t,u,v) - f(t,w,z)| \leq p_{11}(t)|u-w| + p_{12}(t)|v-z|, \ \ t \in [a,b], \ u,v,w,z \in \mathbb{R},$$

and

$$|g(t,u,v) - g(t,w,z)| \leq p_{21}(t)|u-w| + p_{22}(t)|v-z|, \ \ t \in [a,b], \ u,v,w,z \in \mathbb{R}.$$

It is worth mentioning that, for the above fractional differential equation, the method used to obtain the fractional Lyapunov-type inequalities is the Green's function approach that derives the Green's function of the equivalent integral form of the boundary value problem being considered and then finding the maximum or an upper bound of its Green's function. In addition, compared with a large number of references devoted to the study of Lyapunov-type inequalities for fractional differential equations, there is not much undertaken for systems of fractional differential equations. To the best of our knowledge, there is no paper to study the Lyapunov-type inequality for systems of Riemann–Liouville fractional differential equations with coupled boundary conditions. The objective of the present paper is to fill the gap in this area and, more extensively, to study the Lyapunov-type inequalities for the systems of nonlinear fractional differential equations with multi-point coupled boundary conditions (1). Coupled boundary conditions appear in the study of Sturm–Liouville problems and reaction-diffusion Equations [17], and have applications in many fields of sciences and engineering, such as thermal conduction [18] and mathematical biology [19]. The reader may consult the paper [20] for the initial study of differential equations under coupled boundary conditions and the paper [21,22] for fractional differential equations equipped with multi-point boundary conditions.

This paper has three significant features: the boundary conditions contain coupled multi-point boundary conditions and uncoupled multi-point boundary conditions; the establishment of three Lyapunov-type inequalities for (1) is mainly based on matrix analysis and the properties of the operator T defined on two Banach spaces $E \times E$ and $E_1 \times E_2$; the comparison between two Lyapunov-type inequalities for (1) is given under certain special conditions.

The organization of this paper is as follows. In Section 2, we present preliminary definitions and properties of fractional calculus, several essential lemmas associated with this work. The main results and their proof are given in Section 3.

We make the following assumptions on the systems of nonlinear fractional differential equations (1):

(H0) $a_{ij} \geq 0 (i = 1, 2, 3, 4; j = 1, 2, \ldots, n), \kappa_{ij} \geq 0 \, (i, j = 1, 2)$ and $\kappa \stackrel{\Delta}{=} \kappa_{11}\kappa_{22} - \kappa_{12}\kappa_{21} > 0$, where

$$\kappa_{11} = 1 - \sum_{i=1}^{n} \frac{a_{1i}(\xi_i - a)^{\beta_1 - 1}}{(b-a)^{\beta_1 - 1}}, \quad \kappa_{12} = \sum_{j=1}^{n} \frac{a_{2j}(\eta_j - a)^{\beta_2 - 1}}{(b-a)^{\beta_2 - 1}},$$

$$\kappa_{21} = \sum_{i=1}^{n} \frac{a_{3i}(\xi_i - a)^{\beta_1 - 1}}{(b-a)^{\beta_1 - 1}}, \quad \kappa_{22} = 1 - \sum_{j=1}^{n} \frac{a_{4j}(\eta_j - a)^{\beta_2 - 1}}{(b-a)^{\beta_2 - 1}}.$$

(H1) $f, g : [a, b] \times \mathbb{R}^2 \to \mathbb{R}$ are continuous.

(H2) There exist positive functions $p_{11}, p_{12} \in C[a, b]$ such that

$$|f(t, x, y)| \leq p_{11}(t)|x| + p_{12}(t)|y|, \quad t \in [a, b], \; x, y \in \mathbb{R}.$$

(H3) There exist positive functions $p_{21}, p_{22} \in C[a, b]$ such that

$$|g(t, x, y)| \leq p_{21}(t)|x| + p_{22}(t)|y|, \quad t \in [a, b], \; x, y \in \mathbb{R}.$$

2. Preliminaries

In this part, we first give some basic definitions, lemmas and theorems.

Definition 1 ([23,24]). *The derivative with fractional order $\alpha > 0$ of Riemann–Liouville type is defined for the function σ defined on $[a, b]$ as*

$$D^{\alpha}_{a+}\sigma(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^{(n)} \int_a^t \frac{\sigma(s)}{(t-s)^{\alpha+1-n}} ds, \quad t \in [a, b], n = [\alpha] + 1,$$

where $\Gamma(n - \alpha)$ is Euler gamma function.

Definition 2 ([23,24]). *The integral with fractional order $\alpha > 0$ of Riemann–Liouville type is defined for the function σ as*

$$I^{\alpha}_{a+}\sigma(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1}\sigma(s) ds, \quad t \in [a, b],$$

where $\Gamma(\alpha)$ is Euler gamma function.

Lemma 1 ([23,24]). *Suppose that $\varphi \in C[a, b]$, $n \in \mathbb{N}^+$, and $n - 1 < \alpha \leq n$. Thus, the general solution of $D^{\alpha}_{a+}u(t) = \varphi(t)$ is*

$$u(t) = I^{\alpha}_{a+}\varphi(t) + c_1(t-a)^{\alpha-1} + c_2(t-a)^{\alpha-2} + \cdots + c_n(t-a)^{\alpha-n},$$

such that $c_j \in \mathbb{R}$, $j = 1, \ldots, n$.

Let $E = C[a, b]$ be the Banach space equipped with norm $\|x\| = \max_{t \in [a,b]} |x(t)|$ for $x \in E$.

Consider the system of linear fractional differential equations

$$\begin{cases} D^{\beta_1}_{a+}u(t) + \varphi_1(t) = 0, \ \ t \in (a,b), \\ u(a) = 0, \ u(b) = \sum\limits_{i=1}^{n} a_{1i}u(\xi_i) + \sum\limits_{j=1}^{n} a_{2j}v(\eta_j), \\ D^{\beta_2}_{a+}v(t) + \varphi_2(t) = 0, \ \ t \in (a,b), \\ v(a) = 0, \ v(b) = \sum\limits_{i=1}^{n} a_{3i}u(\xi_i) + \sum\limits_{j=1}^{n} a_{4j}v(\eta_j), \end{cases} \tag{6}$$

then one has the following lemma.

Lemma 2. *Let $\varphi_1, \varphi_2 \in C[a,b]$ then (u,v) is a solution of (6) if and only if (u,v) is a solution of the integral equation*

$$\begin{cases} u(t) = \int_a^b G_{11}(t,s)\varphi_1(s)ds + \int_a^b G_{12}(t,s)\varphi_2(s)ds, \\ v(t) = \int_a^b G_{21}(t,s)\varphi_1(s)ds + \int_a^b G_{22}(t,s)\varphi_2(s)ds, \end{cases}$$

where

$$G_{11}(t,s) = G_{\beta_1}(t,s) + \frac{(t-a)^{\beta_1-1}}{\kappa(b-a)^{\beta_1-1}} \sum_{i=1}^{n} (\kappa_{22}a_{1i} + \kappa_{12}a_{3i}) G_{\beta_1}(\xi_i, s),$$

$$G_{12}(t,s) = \frac{(t-a)^{\beta_1-1}}{\kappa(b-a)^{\beta_1-1}} \sum_{j=1}^{n} (\kappa_{22}a_{2j} + \kappa_{12}a_{4j}) G_{\beta_2}(\eta_j, s),$$

$$G_{21}(t,s) = \frac{(t-a)^{\beta_2-1}}{\kappa(b-a)^{\beta_2-1}} \sum_{i=1}^{n} (\kappa_{21}a_{1i} + \kappa_{11}a_{3i}) G_{\beta_1}(\xi_i, s),$$

$$G_{22}(t,s) = G_{\beta_2}(t,s) + \frac{(t-a)^{\beta_2-1}}{\kappa(b-a)^{\beta_2-1}} \sum_{j=1}^{n} (\kappa_{21}a_{2j} + \kappa_{11}a_{4j}) G_{\beta_2}(\eta_j, s),$$

and

$$G_{\beta_i}(t,s) = \frac{1}{\Gamma(\beta_i)} \begin{cases} \frac{(t-a)^{\beta_i-1}}{(b-a)^{\beta_i-1}}(b-s)^{\beta_i-1} - (t-s)^{\beta_i-1}, & a \le s \le t \le b; \\ \frac{(t-a)^{\beta_i-1}}{(b-a)^{\beta_i-1}}(b-s)^{\beta_i-1}, & a \le t \le s \le b. \end{cases} \tag{7}$$

Proof. Deduced from Lemma 1, we obtain

$$\begin{aligned} u(t) &= -I^{\beta_1}_{a+}\varphi_1(t) + c_1(t-a)^{\beta_1-1} + c_2(t-a)^{\beta_1-2}, \\ v(t) &= -I^{\beta_2}_{a+}\varphi_2(t) + d_1(t-a)^{\beta_2-1} + d_2(t-a)^{\beta_2-2}. \end{aligned} \tag{8}$$

Therefore, the general solution of (6) is

$$u(t) = -\int_a^t \frac{(t-s)^{\beta_1-1}}{\Gamma(\beta_1)}\varphi_1(s)ds + c_1(t-a)^{\beta_1-1} + c_2(t-a)^{\beta_1-2},$$

$$v(t) = -\int_a^t \frac{(t-s)^{\beta_2-1}}{\Gamma(\beta_2)}\varphi_2(s)ds + d_1(t-a)^{\beta_2-1} + d_2(t-a)^{\beta_2-2}.$$

Since $u(a) = v(a) = 0$, it is clear that $c_2 = d_2 = 0$. Let $u(b) = c$ and $v(b) = d$, so we conclude that

$$c = u(b) = -\int_a^b \frac{(b-s)^{\beta_1-1}}{\Gamma(\beta_1)}\varphi_1(s)ds + c_1(b-a)^{\beta_1-1},$$
$$d = v(b) = -\int_a^b \frac{(b-s)^{\beta_2-1}}{\Gamma(\beta_2)}\varphi_2(s)ds + d_1(b-a)^{\beta_2-1}.$$

Hence, (8) implies

$$\begin{aligned} u(t) &= \int_a^b G_{\beta_1}(t,s)\varphi_1(s)ds + c\frac{(t-a)^{\beta_1-1}}{(b-a)^{\beta_1-1}}, \\ v(t) &= \int_a^b G_{\beta_2}(t,s)\varphi_2(s)ds + d\frac{(t-a)^{\beta_2-1}}{(b-a)^{\beta_2-1}}, \end{aligned} \tag{9}$$

where $G_{\beta_i}(t,s)$ is given by (7).

It is worth noting that $u(1) = c$ and $v(1) = d$. In order to determine c, d, we require that the function in (9) should satisfy multi-point boundary conditions in (6), i.e.,

$$\begin{aligned} c = u(1) &= \sum_{i=1}^n a_{1i}\int_a^b G_{\beta_1}(\xi_i,s)\varphi_1(s)ds + \sum_{j=1}^n a_{2j}\int_a^b G_{\beta_2}(\eta_j,s)\varphi_2(s)ds \\ &\quad + c\sum_{i=1}^n a_{1i}\frac{(\xi_i-a)^{\beta_1-1}}{(b-a)^{\beta_1-1}} + d\sum_{j=1}^n a_{2j}\frac{(\eta_j-a)^{\beta_2-1}}{(b-a)^{\beta_2-1}}, \\ d = v(1) &= \sum_{i=1}^n a_{3i}\int_a^b G_{\beta_1}(\xi_i,s)\varphi_1(s)ds + \sum_{j=1}^n a_{4j}\int_a^b G_{\beta_2}(\eta_j,s)\varphi_2(s)ds \\ &\quad + c\frac{(\xi_i-a)^{\beta_1-1}}{(b-a)^{\beta_1-1}}\sum_{i=1}^n a_{3i} + d\sum_{j=1}^n a_{4j}\frac{(\eta_j-a)^{\beta_2-1}}{(b-a)^{\beta_2-1}}. \end{aligned}$$

The above two equations are written in matrix form

$$\begin{pmatrix} \kappa_{11} & -\kappa_{12} \\ -\kappa_{21} & \kappa_{22} \end{pmatrix}\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^n a_{1i}\int_a^b G_{\beta_1}(\xi_i,s)\varphi_1(s)ds + \sum_{j=1}^n a_{2j}\int_a^b G_{\beta_2}(\eta_j,s)\varphi_2(s)ds \\ \sum_{i=1}^n a_{3i}\int_a^b G_{\beta_1}(\xi_i,s)\varphi_1(s)ds + \sum_{j=1}^n a_{4j}\int_a^b G_{\beta_2}(\eta_j,s)\varphi_2(s)ds \end{pmatrix}.$$

Since $\kappa = \kappa_{11}\kappa_{22} - \kappa_{12}\kappa_{21} \neq 0$, there is

$$\begin{aligned} c = &\frac{\kappa_{22}}{\kappa}\sum_{i=1}^n a_{1i}\int_a^b G_{\beta_1}(\xi_i,s)\varphi_1(s)ds + \frac{\kappa_{12}}{\kappa}\sum_{i=1}^n a_{3i}\int_a^b G_{\beta_1}(\xi_i,s)\varphi_1(s)ds \\ &+\frac{\kappa_{22}}{\kappa}\sum_{j=1}^n a_{2j}\int_a^b G_{\beta_2}(\eta_j,s)\varphi_2(s)ds + \frac{\kappa_{12}}{\kappa}\sum_{j=1}^n a_{4j}\int_a^b G_{\beta_2}(\eta_j,s)\varphi_2(s)ds \\ = &\frac{1}{\kappa}\sum_{i=1}^n(\kappa_{22}a_{1i} + \kappa_{12}a_{3i})\int_a^b G_{\beta_1}(\xi_i,s)\varphi_1(s)ds \\ &+\frac{1}{\kappa}\sum_{j=1}^n(\kappa_{22}a_{2j} + \kappa_{12}a_{4j})\int_a^b G_{\beta_2}(\eta_j,s)\varphi_2(s)ds, \end{aligned}$$

and

$$\begin{aligned} d = & \frac{\kappa_{21}}{\kappa}\sum_{i=1}^{n} a_{1i}\int_a^b G_{\beta_1}(\xi_i,s)\varphi_1(s)ds + \frac{\kappa_{11}}{\kappa}\sum_{i=1}^{n} a_{3i}\int_a^b G_{\beta_1}(\xi_i,s)\varphi_1(s)ds \\ & + \frac{\kappa_{21}}{\kappa}\sum_{j=1}^{n} a_{2j}\int_a^b G_{\beta_2}(\eta_j,s)\varphi_2(s)ds + \frac{\kappa_{11}}{\kappa}\sum_{j=1}^{n} a_{4j}\int_a^b G_{\beta_2}(\eta_j,s)\varphi_2(s)ds \\ = & \frac{1}{\kappa}\sum_{i=1}^{n}(\kappa_{21}a_{1i}+\kappa_{11}a_{3i})\int_a^b G_{\beta_1}(\xi_i,s)\varphi_1(s)ds \\ & + \frac{1}{\kappa}\sum_{j=1}^{n}(\kappa_{21}a_{2j}+\kappa_{11}a_{4j})\int_a^b G_{\beta_2}(\eta_j,s)\varphi_2(s)ds. \end{aligned}$$

Hence,

$$\begin{aligned} u(t) = & \int_a^b G_{\beta_1}(t,s)\varphi_1(s)ds + \frac{(t-a)^{\beta_1-1}}{\kappa(b-a)^{\beta_1-1}}\sum_{i=1}^{n}(\kappa_{22}a_{1i}+\kappa_{12}a_{3i})\int_a^b G_{\beta_1}(\xi_i,s)\varphi_1(s)ds \\ & + \frac{(t-a)^{\beta_1-1}}{\kappa(b-a)^{\beta_1-1}}\sum_{j=1}^{n}(\kappa_{22}a_{2j}+\kappa_{12}a_{4j})\int_a^b G_{\beta_2}(\eta_j,s)\varphi_2(s)ds \\ = & \int_a^b G_{11}(t,s)\varphi_1(s)ds + \int_a^b G_{12}(t,s)\varphi_2(s)ds, \\ v(t) = & \int_a^b G_{\beta_2}(t,s)\varphi_2(s)ds + \frac{(t-a)^{\beta_2-1}}{\kappa(b-a)^{\beta_2-1}}\sum_{i=1}^{n}(\kappa_{21}a_{1i}+\kappa_{11}a_{3i})\int_a^b G_{\beta_1}(\xi_i,s)\varphi_1(s)ds \\ & + \frac{(t-a)^{\beta_2-1}}{\kappa(b-a)^{\beta_2-1}}\sum_{j=1}^{n}(\kappa_{21}a_{2j}+\kappa_{11}a_{4j})\int_a^b G_{\beta_2}(\eta_j,s)\varphi_2(s)ds \\ = & \int_a^b G_{21}(t,s)\varphi_1(s)ds + \int_a^b G_{22}(t,s)\varphi_2(s)ds \end{aligned}$$

and the proof is complete. □

Lemmas 3–5 below give some important properties of $G_{\beta_i}(t,s)$ and $G_{ij}(t,s)$. Parts (1)–(3) of Lemma 3 are taken from [6] and part (4) of Lemma 3 follows from the expression of $G_{\beta_i}(t,s)$.

Lemma 3 ([6]). *The Green function $G_{\beta_i}(t,s)$ defined above satisfies the following conditions:*

(1) $G_{\beta_i}(t,s) \geq 0$ for all $t,s \in [a,b]$.

(2) $\max\limits_{t\in[a,b]} G_{\beta_i}(t,s) = G_{\beta_i}(s,s)$, $s \in [a,b]$.

(3) $G_{\beta_i}(s,s)$ has a unique maximum, given by

$$\max_{s\in[a,b]} G_{\beta_i}(s,s) = G_{\beta_i}\left(\frac{a+b}{2},\frac{a+b}{2}\right) = \frac{1}{\Gamma(\beta_i)}\left(\frac{b-a}{4}\right)^{\beta_i-1}.$$

(4) $G_{\beta_i}(t,s) \leq \dfrac{1}{\Gamma(\beta_i)}\dfrac{(t-a)^{\beta_i-1}}{(b-a)^{\beta_i-1}}(b-s)^{\beta_i-1}$ for all $t,s \in [a,b]$.

Lemma 4. *For $a < \xi_i < b$, we have*

$$\max_{s\in[a,b]} G_{\beta_i}(\xi_i,s) = G_{\beta_i}(\xi_i,\xi_i) = \frac{1}{\Gamma(\beta_i)}\frac{(\xi_i-a)^{\beta_i-1}}{(b-a)^{\beta_i-1}}(b-\xi_i)^{\beta_i-1}.$$

Proof. Let us start to define two functions

$$g_1(s) = \frac{(\xi_i-a)^{\beta_i-1}}{(b-a)^{\beta_i-1}}(b-s)^{\beta_i-1} - (\xi_i-s)^{\beta_i-1}, \quad a \leq s \leq \xi_i < b,$$

and

$$g_2(s) = \frac{(\xi_i - a)^{\beta_i - 1}}{(b-a)^{\beta_i - 1}}(b-s)^{\beta_i - 1}, \ \ a < \xi_i \leq s \leq b.$$

It is easy to see that $g_2(s)$ is a decreasing function on $[\xi_i, b]$, and we have

$$g_2(s) \leq g_2(\xi_i) = G_{\beta_i}(\xi_i, \xi_i) = \frac{(\xi_i - a)^{\beta_i - 1}}{(b-a)^{\beta_i - 1}}(b-s)^{\beta_i - 1}, \ s \in [\xi_i, b].$$

On the other hand, since $1 < \beta_i \leq 2$ and $s \in [a, \xi_i]$, we have

$$\begin{aligned} \frac{dg_1(s)}{ds} &= -(\beta_i - 1)\left(\frac{(\xi_i - a)^{\beta_i - 1}}{(b-a)^{\beta_i - 1}}(b-s)^{\beta_i - 2} - (\xi_i - s)^{\beta_i - 2}\right) \\ &= -(\beta_i - 1)(b-s)^{\beta_i - 2}\left(\frac{(\xi_i - a)^{\beta_i - 1}}{(b-a)^{\beta_i - 1}} - \frac{(\xi_i - s)^{\beta_i - 2}}{(b-s)^{\beta_i - 2}}\right) \geq 0. \end{aligned}$$

Therefore, $g_1(s)$ is an increasing function on $[a, \xi_i]$ and $g_1(s) \leq g_1(\xi_i) = G_{\beta_i}(\xi_i, \xi_i)$. □

Lemma 5. *Four functions G_{ij} $(i, j = 1, 2)$ defined in Lemma 2 satisfy the following conditions:*
(i) $G_{ij}(t, s) \leq \lambda_{ij}$ for all $a \leq t, s \leq b$,
(ii) $G_{ij}(t, s) \leq \mu_{ij}(t-a)^{\beta_i - 1}(b-s)^{\beta_j - 1}$ for all $a \leq t, s \leq b$,
where λ_{ij}, μ_{ij} $(i, j = 1, 2)$ are given by

$$\lambda_{11} = \frac{1}{\Gamma(\beta_1)}\left(\frac{b-a}{4}\right)^{\beta_1 - 1} + \frac{1}{\kappa\Gamma(\beta_1)}\sum_{i=1}^{n}\frac{(\kappa_{22}a_{1i} + \kappa_{12}a_{3i})(\xi_i - a)^{\beta_1 - 1}}{(b-a)^{\beta_1 - 1}}(b - \xi_i)^{\beta_1 - 1},$$

$$\lambda_{12} = \frac{1}{\kappa\Gamma(\beta_2)(b-a)^{\beta_1 - 1}}\sum_{j=1}^{n}(\kappa_{22}a_{2j} + \kappa_{12}a_{4j})(\eta_j - a)^{\beta_2 - 1}(b - \eta_j)^{\beta_2 - 1},$$

$$\lambda_{21} = \frac{1}{\kappa\Gamma(\beta_1)(b-a)^{\beta_2 - 1}}\sum_{i=1}^{n}(\kappa_{21}a_{1i} + \kappa_{11}a_{3i})(\xi_i - a)^{\beta_1 - 1}(b - \xi_i)^{\beta_1 - 1},$$

$$\lambda_{22} = \frac{1}{\Gamma(\beta_2)}\left(\frac{b-a}{4}\right)^{\beta_2 - 1} + \frac{1}{\kappa\Gamma(\beta_2)}\sum_{j=1}^{n}\frac{(\kappa_{21}a_{2j} + \kappa_{11}a_{4j})(\eta_j - a)^{\beta_2 - 1}}{(b-a)^{\beta_2 - 1}}(b - \eta_j)^{\beta_2 - 1},$$

$$\mu_{11} = \frac{\kappa_{22}}{\kappa\Gamma(\beta_1)(b-a)^{\beta_1 - 1}}, \ \ \mu_{12} = \frac{1}{\kappa\Gamma(\beta_2)(b-a)^{\beta_1 - 1}},$$

and

$$\mu_{21} == \frac{1}{\kappa\Gamma(\beta_1)(b-a)^{\beta_2 - 1}}, \ \ \mu_{22} = \frac{\kappa_{11}}{\kappa\Gamma(\beta_2)(b-a)^{\beta_2 - 1}}.$$

Proof. For sake of simplicity, we only prove Lemma 5 for function $G_{11}(t, s)$. Similar arguments apply for the other function.

Using Lemmas 3 and 4, we obtain

$$\begin{aligned} G_{11}(t,s) &= G_{\beta_1}(t,s) + \frac{(t-a)^{\beta_1 - 1}}{\kappa(b-a)^{\beta_1 - 1}}\sum_{i=1}^{n}(\kappa_{22}a_{1i} + \kappa_{12}a_{3i})G_{\beta_1}(\xi_i, s) \\ &\leq G_{\beta_1}(s,s) + \frac{1}{\kappa}\sum_{i=1}^{n}(\kappa_{22}a_{1i} + \kappa_{12}a_{3i})G_{\beta_1}(\xi_i, \xi_i) \\ &\leq \frac{1}{\Gamma(\beta_1)}\left(\frac{b-a}{4}\right)^{\beta_1 - 1} + \frac{1}{\kappa\Gamma(\beta_1)}\sum_{i=1}^{n}\frac{(\kappa_{22}a_{1i} + \kappa_{12}a_{3i})(\xi_i - a)^{\beta_1 - 1}}{(b-a)^{\beta_1 - 1}}(b - \xi_i)^{\beta_1 - 1} \\ &= \lambda_{11}, \end{aligned}$$

and

$$\begin{aligned}
G_{11}(t,s) &= G_{\beta_1}(t,s) + \frac{(t-a)^{\beta_1-1}}{\kappa(b-a)^{\beta_1-1}} \sum_{i=1}^{n} (\kappa_{22}a_{1i} + \kappa_{12}a_{3i}) G_{\beta_1}(\xi_i, s) \\
&\leq \frac{1}{\Gamma(\beta_1)} \frac{(t-a)^{\beta_1-1}}{(b-a)^{\beta_1-1}} (b-s)^{\beta_1-1} \\
&\quad + \frac{(t-a)^{\beta_1-1}}{\kappa(b-a)^{\beta_1-1}} \sum_{i=1}^{n} (\kappa_{22}a_{1i} + \kappa_{12}a_{3i}) \frac{1}{\Gamma(\beta_1)} \frac{(\xi_i-a)^{\beta_1-1}}{(b-a)^{\beta_1-1}} (b-s)^{\beta_1-1} \\
&= \frac{1}{\Gamma(\beta_1)} \frac{(t-a)^{\beta_1-1}}{(b-a)^{\beta_1-1}} (b-s)^{\beta_1-1} \left[1 + \frac{\kappa_{22} \sum_{i=1}^{n} a_{1i} + \kappa_{12} \sum_{i=1}^{n} a_{3i}}{\kappa(b-a)^{\beta_1-1}} (\xi_i - a)^{\beta_1-1} \right] \\
&= \frac{1}{\Gamma(\beta_1)} \frac{(t-a)^{\beta_1-1}}{(b-a)^{\beta_1-1}} (b-s)^{\beta_1-1} \left[1 + \frac{\kappa_{22}(1-\kappa_{11}) + \kappa_{12}\kappa_{21}}{\kappa} \right] \\
&= \frac{\kappa_{22}}{\kappa\Gamma(\beta_1)} \frac{(t-a)^{\beta_1-1}}{(b-a)^{\beta_1-1}} (b-s)^{\beta_1-1} = \mu_{11}(t-a)^{\beta_1-1}(b-s)^{\beta_1-1}.
\end{aligned}$$

□

In the next section, nonnegative square matrices will be used in order to present Lyapunov-type inequalities for systems of nonlinear fractional differential Equation (1). In this proof, a key role will be played by the so-called convergent to zero matrices. A nonnegative square matrix M is said to be convergent to zero if

$$M^k \to 0, \text{ as } k \to \infty.$$

Let us recall that a real square matrix $M = (b_{ij})_{2\times 2}$ is said to be nonnegative and write $M \geq 0$ if $b_{ij} \geq 0$ for $i, j = 1, 2$. For square matrix M_1, M_2, we say $M_1 \geq M_2$ if $M_1 - M_2 \geq 0$. Similar definitions and notation apply for vectors.

We denote by M_2^+ the set of square nonnegative matrices. For a matrix $M \in M_2^+$, let Trace(M), det(M) and $\rho(M)$ denote the trace, the determinant and the spectral radius of M, respectively.

Lemma 6 ([25]). *Let $C \in M_2^+$. If $\rho(C) < 1$, then*

$$\lim_{n\to\infty} C^n = 0.$$

Lemma 7 ([16]). *Let $C \in M_2^+$. Then*

$$\rho(C) = \frac{Trace(C) + \sqrt{[Trace(C)]^2 - 4det(C)}}{2}.$$

Lemma 8 ([26]). *If $M_1 \geq M_2 \geq 0$, then $\rho(M_2) \leq \rho(M_1)$.*

3. Main Results

For $p_{ij} \in C[a,b]$ $(i, j = 1, 2)$, let

$$J_{ij}(p_{1j}, p_{2j}) = \lambda_{i1} \int_a^b p_{1j}(s)ds + \lambda_{i2} \int_a^b p_{2j}(s)ds, \quad i, j = 1, 2.$$

Theorem 1. *Suppose that (H0)-(H3) are satisfied. If (1) have a nontrivial solution, then*

$$\begin{aligned}
&J_{11}(p_{11}, p_{21}) + J_{22}(p_{12}, p_{22}) \\
&+ \sqrt{[J_{11}(p_{11}, p_{21}) - J_{22}(p_{12}, p_{22})]^2 + 4J_{12}(p_{12}, p_{22})J_{21}(p_{11}, p_{21})} \geq 2.
\end{aligned} \tag{10}$$

Proof. Let $(u^*, v^*) \in E \times E$ be a nontrivial solution of (1), and suppose that

$$\begin{aligned} &J_{11}(p_{11}, p_{21}) + J_{22}(p_{12}, p_{22}) \\ &+\sqrt{[J_{11}(p_{11}, p_{21}) - J_{22}(p_{12}, p_{22})]^2 + 4J_{12}(p_{12}, p_{22})J_{21}(p_{11}, p_{21})} < 2. \end{aligned} \tag{11}$$

Let us introduce the operator $T : E \times E \to E \times E$ given by

$$T(u, v) = (T_1(u, v), T_2(u, v)), \quad u, v \in E,$$

where

$$T_1(u, v)(t) = \int_a^b G_{11}(t, s) f(s, u(s), v(s))ds + \int_a^b G_{12}(t, s) g(s, u(s), v(s))ds,$$

$$T_2(u, v)(t) = \int_a^b G_{21}(t, s) f(s, u(s), v(s))ds + \int_a^b G_{22}(t, s) g(s, u(s), v(s))ds,$$

and G_{ij} $(i, j = 1, 2)$ is defined in Lemma 2. By Lemma 2, (u^*, v^*) is a nontrivial fixed point of T.

Using (H2) and Lemma 5, for all $t \in [0, 1]$, we obtain

$$\begin{aligned} |u^*(t)| &= |T_1(u^*, v^*)(t)| \\ &\leq \int_a^b G_{11}(t, s)|f(s, u^*(s), v^*(s))|ds + \int_a^b G_{12}(t, s)|g(s, u^*(s), v^*(s))|ds \\ &\leq \int_a^b G_{11}(t, s)(p_{11}(s)|u^*(s)| + p_{12}(s)|v^*(s)|)ds \\ &\quad + \int_a^b G_{12}(t, s)(p_{21}(s)|u^*(s)| + p_{22}(s)|v^*(s)|)ds \\ &\leq \left(\lambda_{11} \int_a^b p_{11}(s)ds + \lambda_{12} \int_a^b p_{21}(s)ds\right) \|u^*\| \\ &\quad + \left(\lambda_{11} \int_a^b p_{12}(s)ds + \lambda_{12} \int_a^b p_{22}(s)ds\right) \|v^*\|. \end{aligned}$$

Therefore, we obtain

$$\|u^*\| \leq J_{11}(p_{11}, p_{21})\|u^*\| + J_{12}(p_{12}, p_{22})\|v^*\|. \tag{12}$$

Similarly, using Lemma 5 and (H3), we have

$$\|v^*\| \leq J_{21}(p_{11}, p_{21})\|u^*\| + J_{22}(p_{12}, p_{22})\|v^*\|. \tag{13}$$

Combining (12) with (13), we deduce that

$$\begin{pmatrix} \|u^*\| \\ \|v^*\| \end{pmatrix} \leq \begin{pmatrix} J_{11}(p_{11}, p_{21}) & J_{12}(p_{12}, p_{22}) \\ J_{21}(p_{11}, p_{21}) & J_{22}(p_{12}, p_{22}) \end{pmatrix} \begin{pmatrix} \|u^*\| \\ \|v^*\| \end{pmatrix}.$$

By induction, for $n \in \mathbb{N}$, we have

$$\begin{pmatrix} \|u^*\| \\ \|v^*\| \end{pmatrix} \leq \begin{pmatrix} J_{11}(p_{11}, p_{21}) & J_{12}(p_{12}, p_{22}) \\ J_{21}(p_{11}, p_{21}) & J_{22}(p_{12}, p_{22}) \end{pmatrix}^n \begin{pmatrix} \|u^*\| \\ \|v^*\| \end{pmatrix}.$$

Next, using Lemma 6, Lemma 7 and (11), we deduce that

$$\|u^*\| = \|v^*\| = 0,$$

which contradicts the nontriviality of (u^*, v^*). This proves (10). □

For $p_{ij} \in C[a,b]$ $(i,j=1,2)$, let

$$\begin{aligned} I_{ij}(p_{1j},p_{2j}) &= \mu_{i1}\int_a^b p_{1j}(s)(b-s)^{\beta_1-1}(s-a)^{\beta_j-1}ds \\ &\quad +\mu_{i2}\int_a^b p_{2j}(s)(b-s)^{\beta_2-1}(s-a)^{\beta_j-1}ds, \ \ i,j=1,2. \end{aligned}$$

Theorem 2. *Suppose that (H0)-(H3) are satisfied. If (1) have a nontrivial solution, then*

$$\begin{aligned} &I_{11}(p_{11},p_{21})+I_{22}(p_{12},p_{22}) \\ &+\sqrt{[I_{11}(p_{11},p_{21})-I_{22}(p_{12},p_{22})]^2+4I_{12}(p_{12},p_{22})I_{21}(p_{11},p_{21})} \geq 2. \end{aligned} \tag{14}$$

Proof. Let $(u^*,v^*) \in E\times E$ be a nontrivial solution of (1), and suppose that

$$\begin{aligned} &I_{11}(p_{11},p_{21})+I_{22}(p_{12},p_{22}) \\ &+\sqrt{[I_{11}(p_{11},p_{21})-I_{22}(p_{12},p_{22})]^2+4I_{12}(p_{12},p_{22})I_{21}(p_{11},p_{21})} < 2. \end{aligned} \tag{15}$$

For $u,v \in E$, by the bounded property of continuous functions, there is $K>0$ such that

$$|f(s,u(s),v(s))| \leq K, \ |g(s,u(s),v(s))| \leq K, \ \ s\in[a,b].$$

With the use of Lemma 5, we have

$$\begin{aligned} |T_1(u,v)(t)| &\leq \int_a^b G_{11}(t,s)|f(s,u(s),v(s))|ds+\int_a^b G_{12}(t,s)|g(s,u(s),v(s))|ds \\ &\leq K\mu_{11}(t-a)^{\beta_1-1}\int_a^b (b-s)^{\beta_1-1}ds+K\mu_{12}(t-a)^{\beta_1-1}\int_a^a (b-s)^{\beta_2-1}ds \\ &= \left(\frac{K\mu_{11}(b-a)^{\beta_1}}{\beta_1}+\frac{K\mu_{12}(b-a)^{\beta_2}}{\beta_2}\right)(t-a)^{\beta_1-1}, \end{aligned}$$

and

$$|T_2(u,v)(t)| \leq \left(\frac{K\mu_{21}(b-a)^{\beta_1}}{\beta_1}+\frac{K\mu_{22}(b-a)^{\beta_2}}{\beta_2}\right)(t-a)^{\beta_2-1}.$$

This implies that T_i maps all of $E\times E$ into the vector subspace E_i of E, where E_i is given by

$$E_i=\{u\in E: \ \text{ there is } M>0 \text{ such that } |u(t)| \leq M(t-a)^{\beta_i-1}, \ t\in[a,b]\}.$$

Evidently, E_i $(i=1,2)$ are Banach spaces with the norm

$$\|u\|_i=\inf\{M>0: \ |u(t)| \leq M(t-a)^{\beta_i-1}, \ t\in[a,b]\}.$$

Therefore, (u^*,v^*) is a nontrivial fixed point of T in $E_1\times E_2$.

Using (H2) and Lemma 5, for all $t \in [0,1]$, we obtain

$$\begin{aligned}
&|u(t)| \\
\leq\ & \int_a^b G_{11}(t,s)|f(s,u(s),v(s))|ds + \int_a^b G_{12}(t,s)|g(s,u(s),v(s))|ds \\
\leq\ & \int_a^b G_{11}(t,s)(p_{11}(s)|u(s)| + p_{12}(s)|v(s)|)ds \\
\leq\ & \int_a^b G_{12}(t,s)(p_{21}(s)|u(s)| + p_{22}(s)|v(s)|)ds \\
\leq\ & \mu_{11}(t-a)^{\beta_1-1}\int_a^b (b-s)^{\beta_1-1}(p_{11}(s)\|u\|_1(s-a)^{\beta_1-1} + p_{12}(s)\|v\|_2(s-a)^{\beta_2-1})ds \\
& +\mu_{12}(t-a)^{\beta_1-1}\int_a^b (b-s)^{\beta_2-1}(p_{21}(s)\|u\|_1(s-a)^{\beta_1-1} + p_{22}(s)\|v\|_2(s-a)^{\beta_2-1})ds \\
\leq\ & \left(\mu_{11}\int_a^b p_{11}(s)(b-s)^{\beta_1-1}(s-a)^{\beta_1-1}ds\right. \\
& \left.+\mu_{12}\int_a^b p_{21}(s)(b-s)^{\beta_2-1}(s-a)^{\beta_1-1}ds\right)\|u\|_1(t-a)^{\beta_1-1} \\
& +\left(\mu_{11}\int_a^b p_{12}(s)(b-s)^{\beta_1-1}(s-a)^{\beta_2-1}ds\right. \\
& \left.+\mu_{12}\int_a^b p_{22}(s)(b-s)^{\beta_2-1}(s-a)^{\beta_2-1}ds\right)\|v\|_2(t-a)^{\beta_1-1}.
\end{aligned}$$

Therefore, we obtain

$$\|u\|_1 \leq I_{11}(p_{11},p_{21})\|u\|_1 + I_{12}(p_{12},p_{22})\|v\|_2. \tag{16}$$

Similarly, using Lemma 5 and (H3), we conclude

$$\|v\|_1 \leq I_{21}(p_{11},p_{21})\|u\|_1 + I_{22}(p_{12},p_{22})\|v\|_2. \tag{17}$$

Combining (16) with (18), we deduce that

$$\begin{pmatrix} \|u\|_1 \\ \|v\|_2 \end{pmatrix} \leq \begin{pmatrix} I_{11}(p_{11},p_{21}) & I_{12}(p_{12},p_{22}) \\ I_{21}(p_{11},p_{21}) & I_{22}(p_{12},p_{22}) \end{pmatrix}\begin{pmatrix} \|u\|_1 \\ \|v\|_2 \end{pmatrix}.$$

With the consideration of Lemma 6, Lemma 7 and (15), we deduce that

$$\|u^*\|_1 = \|v^*\|_2 = 0,$$

which contradicts the nontriviality of (u^*, v^*). This proves (14). □

Let

$$g_{ij}(s) = (b-s)^{\beta_i-1}(s-a)^{\beta_j-1},\ \ s \in [a,b],\ \ i,j = 1,2.$$

Now, we differentiate $g_{ij}(s)$ on (a,b) and we obtain

$$g'_{ij}(s) = (b-s)^{\beta_i-2}(s-a)^{\beta_j-2}[a(\beta_i-1) + b(\beta_j-1) - s(\beta_i+\beta_j-2)]$$

which implies that $g'_3(s) = 0$ only at $s^*_{ij} = \frac{b(\beta_j-1)+a(\beta_i-1)}{\beta_i+\beta_j-2}$. Note that $g_{ij}(a) = g_{ij}(b) = 0$. By the continuity of g_{ij}, we conclude that

$$g_{ij}(s) \leq g_{ij}(s^*_{ij}) = \frac{(b-a)^{\beta_i+\beta_j-2}(\beta_i-1)^{\beta_i-1}(\beta_j-1)^{\beta_j-1}}{(\beta_i+\beta_j-2)^{\beta_i+\beta_j-2}}. \tag{18}$$

For $p_{ij} \in C[a,b]$ $(i,j=1,2)$, let

$$\zeta_{ij}(p_{1j},p_{2j}) = \mu_{i1}g_{1j}(s_{1j}^*)\int_a^b p_{1j}(s)ds + \mu_{i2}g_{2j}(s_{2j}^*)\int_a^b p_{2j}(s)ds,\ \ i,j=1,2.$$

The following theorems are immediate.

Theorem 3. *Suppose that (H0)-(H3) are satisfied. If (1) have a nontrivial solution, then*

$$\begin{aligned}&\zeta_{11}(p_{11},p_{21}) + \zeta_{22}(p_{12},p_{22})\\ &+\sqrt{[\zeta_{11}(p_{11},p_{21}) - \zeta_{22}(p_{12},p_{22})]^2 + 4\zeta_{12}(p_{12},p_{22})\zeta_{21}(p_{11},p_{21})} \geq 2.\end{aligned} \tag{19}$$

Finally, we will compare two inequalities (10) and (19) in the case that $\beta_1 = \beta_2$. If $\beta_1 = \beta_2$, by (18), we obtain

$$g(s^*) \triangleq g_{ij}(s_{ij}^*) = \left(\frac{b-a}{2}\right)^{2\beta_1-2},\ \ i,j=1,2, \tag{20}$$

and square nonnegative matrices $M_1 = (I_{ij}(p_{1j},p_{2j}))_{2\times 2}$ and $M_2 = (\zeta_{ij}(p_{1j},p_{2j}))_{2\times 2}$ become

$$M_1 = \begin{pmatrix} \lambda_{11} & \lambda_{12} \\ \lambda_{21} & \lambda_{22} \end{pmatrix}\begin{pmatrix} \int_a^b p_{11}(s)ds & \int_a^b p_{12}(s)ds \\ \int_a^b p_{21}(s)ds & \int_a^b p_{22}(s)ds \end{pmatrix},$$

and

$$M_2 = g(s^*)\begin{pmatrix} \mu_{11} & \mu_{12} \\ \mu_{21} & \mu_{22} \end{pmatrix}\begin{pmatrix} \int_a^b p_{11}(s)ds & \int_a^b p_{12}(s)ds \\ \int_a^b p_{21}(s)ds & \int_a^b p_{22}(s)ds \end{pmatrix}.$$

It is worth noting that

$$\lambda_{11} = \frac{1}{\Gamma(\beta_1)}\left(\frac{b-a}{4}\right)^{\beta_1-1} + \frac{1}{\kappa\Gamma(\beta_1)}\sum_{i=1}^{n}(\kappa_{22}a_{1i} + \kappa_{12}a_{3i})(\xi_i - a)^{\beta_1-1}\frac{(b-\xi_i)^{\beta_1-1}}{(b-a)^{\beta_1-1}},$$

and

$$\mu_{11}g(s^*) = \frac{1}{\Gamma(\beta_1)}\left(\frac{b-a}{4}\right)^{\beta_1-1} + \frac{1}{\kappa\Gamma(\beta_1)}\sum_{i=1}^{n}(\kappa_{22}a_{1i} + \kappa_{12}a_{3i})(\xi_i - a)^{\beta_1-1}\left(\frac{1}{4}\right)^{\beta_1-1}.$$

Then, it is easily seen that $\lambda_{11} \leq \mu_{11}g(s^*)$ if $\frac{a+3b}{4} \leq \xi_i < b$ for all $i = 1,2,\ldots,n$, and $\lambda_{11} \geq \mu_{11}g(s^*)$ if $a < \xi_i \leq \frac{a+3b}{4}$ for all $i = 1,2,\ldots,n$. In the same way, for $i,j = 1,2$, we can prove

(i) $\lambda_{ij} \leq \mu_{ij}g(s^*)$ if $\frac{a+3b}{4} \leq \xi_k, \eta_k < b$ for all $k = 1,2,\ldots,n$,

(ii) $\lambda_{j1} \geq \mu_{j1}g(s^*)$ if $a < \xi_k, \eta_k \leq \frac{a+3b}{4}$ for all $k = 1,2,\ldots,n$.

Therefore, if $\frac{a+3b}{4} \leq \xi_k, \eta_k < b$ for all $k = 1,2,\ldots,n$, we obtain that $0 \leq M_1 \leq M_2$. This together with Lemma 8 show that inequality (10) is the improvement of inequality (19). Similarly, inequality (19) is the improvement of inequality (10) if $a < \xi_k, \eta_k \leq \frac{a+3b}{4}$ for all $k = 1,2,\ldots,n$.

4. Conclusions

In this article, we investigate a system of Riemann–Liouville fractional differential equations with multi-point coupled boundary conditions. The first Lyapunov-type inequality is obtained by matrix analysis and the Green's function approach, where the Green's

function approach implies deriving the Green's function of the equivalent integral form of the boundary value problem being considered and then finding an upper bound of its Green's function (note that the maximum value of the Green's function of (1) exists, but the analysis is somewhat complicated). The second Lyapunov-type inequality is obtained by matrix analysis and the properties of the operator T defined on Banach spaces $E_1 \times E_2$. The third Lyapunov-type inequality is the corollary of the second Lyapunov-type inequality. Finally, the comparison between the first and the third Lyapunov-type inequalities is given under certain special conditions. We expect that the second approach used in this paper can be applied to study a system of various fractional boundary value problems, and we will continue to discuss the optimal version of the constants appearing in the Lyapunov-type inequality and seek other ways to obtain fractional Lyapunov-type inequality in future papers.

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