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Approximate Controllability of Ψ-Hilfer Fractional Neutral Differential Equation with Infinite Delay

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Abstract: In this paper, we explain the approximate controllability of Ψ-Hilfer fractional neutral differential equations with infinite delay. The outcome is demonstrated using the infinitesimal operator, fractional calculus, semigroup theory, and the Krasnoselskii's fixed point theorem. To begin, we emphasise the presence of the mild solution and show that the Ψ-Hilfer fractional system is approximately controllable. Additionally, we present theoretical and practical examples.

Keywords: Ψ-Hilfer fractional derivative; mild solution; fixed point theorem; infinitesimal generator

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1. Introduction

Fractional calculus equations that include not only one but numerous FD_{ve} are highly concentrated in many physical processes. Because of its astounding uses in exhibiting the wonders of science and technology, the FD_{tial} system has recently received a lot of interest in its significance. Numerous problems in a number of domains, such as visco-elasticity, electrical systems, electro-chemistry, fluid flow, and others, can be managed through the use of fractional systems. There are many uses and applications for the extension of differential equations and inequalities called differential inclusions, which may be thought of as an in optimal control theory. Dynamical systems that have velocities that are not just governed by the system's state are simpler to investigate when one is skilled at using differential inclusions. Studies on boundary value issues have been widely conducted. Numerous studies have been conducted to determine whether there are solutions for FD_{tial} systems and whether there are solutions for FD_{tial} inclusions. To validate the discussion of theory and its application connected to fractional calculus, the given research papers in [1–14] can be consulted.

A crucial idea in mathematical control theory, controllability, is significant in both pure and practical mathematics. Nowadays, controllability has an important role in fractional calculus; thus, researchers have much interest in this area and developing a new concept and idea related to control theory, i.e., how to apply control theory in FD_{tial} systems. Recent years have seen several researchers make significant progress in their understanding of the exact and approximate controllability of different types of dynamical systems including delay or not. The research articles in [15–25] can be used to validate discussions of theory and practise connected to controllability.

Recently, generic FD_{ve} have been developed, in particular, ones like the FD_{ve} with respect to another function. Almeida [26] introduced a new form of FD_{ve} in 2017 by taking into account the Caputo fractional derivative (CFD_{ve}) with respect to an additional function Ψ, or Ψ-CFD_{ve}, in order to improve the accuracy of objective modelling. Then, the authors of [27] introduced the so-called Ψ-Hilfer fractional derivative (Ψ-HFD_{ve}), a FD_{ve} with respect to an another function. The benefits of the Ψ-Caputo and Ψ-Hilfer models that are herein proposed include the freedom to select the classical differential operator and the Ψ-function, i.e., from the selection of the Ψ-function, the classical differentiation operator may act on the fractional integral operator, or alternatively, the fractional integral operator may act on the classical differentiation operator. Motivated by these two articles, researchers have studied more about Ψ-Caputo and Ψ-Hilfer and have developed new works. In [28], the authors studied the existence, uniqueness, and stability of different kinds of mild solutions for Ψ-CFD_{tial} systems with an infinitesimal generator, $\mathtt{A}$. In [29], the researcher discussed the existence and uniqueness of Ψ-Hilfer neutral FD_{tial} equations with infinite delay via a fixed point method. Recently, the authors of [30] investigated the stability and controllability of Ψ-HFD_{ve} via fixed point theory and a semigroup approach. This paper is devoted to exploring a new class of Ψ-Hilfer fractional integro-differential systems under the influence of impulses. Moreover, we prove the novel stability criteria for the considered system by using the Grönwall inequality and investigate the controllability results for the proposed system by using the new piecewise control function.

To our knowledge, no article has been published on the approximate controllability of Ψ-HFD_{tial} equations with infinite delay, and also, motivated by the research in the above articles, we study the approximate controllability of the systems, given by the following:

$$\begin{cases} {}^{H}D_{0^+}^{\eta,\xi;\Psi}\left[\mathtt{u}(\varrho) - \mathtt{H}(\rho,\mathtt{u}_\rho)\right] = \mathtt{A}\mathtt{u}(\varrho) + \mathtt{B}v(\varrho) + \mathtt{G}\left(\varrho,\mathtt{u}_\varrho,\int_0^\varrho e(\varrho,s,\mathtt{u}_s)ds\right), & \varrho \in \mathcal{I}' = (0,b], \\ I_{0^+}^{(1-\eta)(1-\xi)}\mathtt{u}(0) = \phi_0 \in L^2(D,\mathtt{S}_w),\ \varrho \in (-\infty,0], \end{cases} \tag{1}$$

where $\mathtt{A}$ is an infinitesimal generator of the analytic semigroup $\{T(\varrho),\ \varrho \geq 0\}$ on Y. $D_{0^+}^{\eta,\xi;\Psi}$ denotes the Ψ-HFD_{ve} of order η, $0 < \eta < 1$ and type ξ, $0 \leq \xi \leq 1$. Let $\mathtt{u}(\cdot)$ be the state in a Banach space Y with norm $\|\cdot\|$ and $v(\cdot)$ be the control function in $L^2(\mathcal{I},U)$, where U be the Banach space. Here, $\mathtt{B}$ is the bounded linear operator from U to Y. Let $\mathcal{I} = [0,b]$, $\mathtt{H} : \mathcal{I} \times \mathtt{S}_w \to Y$ is the neutral function, $\mathtt{G} : \mathcal{I} \times \mathtt{S}_w \times Y \to Y$ be the appropriate function, $e : \mathcal{I} \times \mathcal{I} \times \mathtt{S}_w \to Y$ and $0 < \varrho_1 < \varrho_2 < \cdots < \varrho_n \leq b$, $\xi : \mathtt{S}_w^n \to \mathtt{S}_w$ are the appropriate functions, where $\mathtt{S}_w$ is a phase space. The histories $\mathtt{u}_\varrho : (-\infty,0] \to Y$ such that $\mathtt{u}_\varrho(s) = \mathtt{u}(\varrho+s)$ belong to the phase space, $\mathtt{S}_w$.

The article's structure is broken down as follows: In Section 2, the fundamentals of fractional calculus, Ψ-Hilfer fractional, and semigroup are discussed. We first establish the mild solution's existence in Section 3 before extending to the system's approximate controllability. To illustrate our main points, we give an example in Section 4. A few conclusions are presented towards the end.

2. Preliminaries

Here, we introduce the fundamental terms, theorems, and lemma that are used throughout the whole text. We introduce a new set

$$\mathtt{S} = \left\{\mathtt{u} \in C(\mathcal{I}',Y) : \lim_{\rho\to 0}\left(\Psi(\rho) - \Psi(0)\right)^{(1-\eta)(1-\xi)}\mathtt{u}(\rho) \text{ exists and infinite}\right\}$$

with norm $\|\cdot\|_{\mathtt{S}}$ defined by

$$\left\|\mathtt{u}(\rho)\right\|_{\mathtt{S}} = \sup_{\rho\in\mathcal{I}'}\left|\left(\Psi(\rho) - \Psi(0)\right)^{(1-\eta)(1-\xi)}\mathtt{u}(\rho)\right|$$

where Ψ is an increasing function with $\Psi'(\varrho) \neq 0$, $\forall\ \varrho \in \mathcal{I}$.

Definition 1 ([31]). *Suppose* $\mathtt{G} : [0, \infty) \to \mathbb{R}]$ *is a real valued function, the Laplace transform is represented and presented by*

$$\mathcal{L}\{\mathtt{G}(\varrho)\}(\vartheta) = \mathcal{G}(\vartheta) = \int_0^\infty \mathtt{G}(\varrho)e^{-\vartheta\varrho}d\varrho, \quad for\ \vartheta < 0.$$

Furthermore, if $\mathcal{G}(\vartheta) = \mathcal{L}\{\mathtt{G}(\varrho)\}$ *and* $G(\vartheta) = \mathcal{L}\{g(\varrho)\}$*, then*

$$\mathcal{L}\left\{ \int_0^\varrho \mathtt{G}(\varrho - \tau)g(\tau)d\tau \right\}(\vartheta) = \mathcal{G}(\vartheta)G(\vartheta). \tag{2}$$

Definition 2 ([31]). *The Laplace transform of* $\mathtt{G}$ *with respect to* Ψ *is presented by*

$$\mathcal{L}_\Psi\{\mathtt{G}(\varrho)\}(\vartheta) = \mathtt{G}(\vartheta) = \int_a^\infty \mathtt{G}(\varrho)e^{-\vartheta(\Psi(\varrho)-\Psi(a))}\mathtt{G}(\varrho)\Psi'(\varrho)d\varrho\ for\ all\ \vartheta \in \mathbb{C}. \tag{3}$$

Definition 3 ([27]). *The* Ψ*–Riemann–Liouville fractional integral of order* η *of the function* $\mathtt{G}$ *is presented by*

$$I_{a^+}^{\eta;\Psi}\mathtt{G}(\vartheta) = \frac{1}{\Gamma(\eta)} \int_a^\vartheta \Psi'(\varrho)(\Psi(\vartheta) - \Psi(\varrho))^{\eta-1}\mathtt{G}(\varrho)d\varrho, \tag{4}$$

where $\eta \in (m-1, m)$*.*

Definition 4 ([27]). *The* Ψ*–Riemann–Liouville* FD_{ve} *of order* η *of the function* $\mathtt{G}$ *is presented by*

$$D_{a^+}^{\eta;\Psi}\mathtt{G}(\vartheta) = \left(\frac{1}{\Psi'(\vartheta)}\frac{d}{d\vartheta}\right)^m I^{m-\eta;\Psi}\mathtt{G}(\vartheta) \tag{5}$$

$$= \frac{1}{\Gamma(m-\eta)}\left(\frac{1}{\Psi'(\vartheta)}\frac{d}{d\vartheta}\right)^m \int_a^\vartheta \Psi'(\varrho)(\Psi(\vartheta) - \Psi(\varrho))^{m-\eta-1}\mathtt{G}(\varrho)d\varrho, \tag{6}$$

where $\eta \in (m-1, m)$*.*

Definition 5 ([26]). *The* Ψ*-*CFD_{ve} *of order* η *is defined by*

$$\begin{aligned} {}^C D_{a^+}^{\eta;\Psi}\mathtt{G}(\varrho) &= \left({}_a I_\Psi^{m-\eta}\mathtt{G}^{[m]}(\varrho)\right) \\ &= \frac{1}{\Gamma(m-\eta)} \int_0^\varrho (\Psi(\varrho) - \Psi(\vartheta))^{m-\eta-1}\mathtt{G}^{[m]}(\vartheta)\Psi'(\vartheta)d\vartheta \end{aligned}$$

where $m = [\eta] + 1$ *and* $\mathtt{G}^{[m]}(\varrho) = \left(\frac{1}{\Psi'(\varrho)}\frac{d}{d\varrho}\right)^n\mathtt{G}(\varrho)$ *in* $[a, b]$*.*

Definition 6 ([27]). *The* Ψ*-*HFD_{ve} *of function* $\mathtt{G}$ *of order* η *and type* ξ *is presented by*

$${}^H D_{a^+}^{\eta,\xi;\Psi}\mathtt{G}(\rho) = I_{a^+}^{\xi(m-\eta);\Psi}\left(\frac{1}{\Psi'(\rho)}\frac{d}{d\rho}\right)^m I_{a^+}^{(1-\xi)(n-\eta);\Psi}\mathtt{G}(\rho)$$

Remark 1. *The* Ψ*-*HFD_{ve} *can be written in the following form:*

$${}^H D_{a^+}^{\eta,\xi;\Psi}\mathtt{G}(\rho) = I_{a^+}^{\xi(m-\eta);\Psi} D_{a^+}^{\eta+\xi(n-\eta);\Psi}\mathtt{G}(\rho)$$

and

$${}^H D_{b^-}^{\eta,\xi;\Psi}\mathtt{G}(\rho) = I_{b^-}^{\xi(m-\eta);\Psi} D_{a^+}^{\eta+\xi(n-\eta);\Psi}\mathtt{G}(\rho),$$

where $-\infty \le a < b \le \infty$*.*

Here, we define the weighted space [27]:

$$\mathtt{C}_{\Psi}(\mathcal{I}, Y) = \{\mathtt{u} : [0, b] \to Y : (\psi(\rho) - \psi(\omega))^{(1-\eta)(1-\xi)}\mathtt{u}(\rho) \in C(\mathcal{I}, Y)\}.$$

Ref. [16]. Next, we define the abstract phase space, $\mathtt{S}_w$. Let $w : (-\infty, 0] \to (0, +\infty)$ be continuous along $\Upsilon = \int_{-\infty}^{0} w(\varrho)d\varrho < +\infty$. Then, for every $n > 0$, we have

$$\mathtt{S} = \left\{\delta : [-n, 0] \to Y \ : \ \delta(\varrho) \text{ is bounded and measurable}\right\},$$

and set the space, $\mathtt{S}$, with the norm

$$\|\delta\|_{[-n,0]} = \sup_{\tau\in[-n,0]} \|\delta(\tau)\|, \ \textit{for all } \delta \in \mathtt{S}.$$

Here, we define

$$\mathtt{S}_w = \left\{\delta : (-\infty, 0] \to Y \text{ such that for any } n > 0,\ \delta|_{[-n,0]} \in \mathtt{S} \textit{ and} \right.$$
$$\left. \int_{-\infty}^{0} (\Psi(\tau) - \Psi(0))^{(1-\eta)(1-\xi)} w(\tau)\|\delta\|_{[\tau,0]} d\tau < +\infty \right\}.$$

If $\mathtt{S}_w$ is endowed with

$$\|\delta\|_{\Upsilon} = \int_{-\infty}^{0} (\Psi(\tau) - \Psi(0))^{(1-\eta)(1-\xi)} w(\tau)\|\delta\|_{[\tau,0]} d\tau, \ \textit{for all } \delta \in \mathtt{S}_w,$$

Thus, $(\mathtt{S}_w, \|\cdot\|_{\Upsilon})$ is a Banach space.
Here, we consider the set

$$\mathtt{S}'_w = \left\{\mathtt{u} : (-\infty, b] \to Y \ : \ \mathtt{u} \in \mathtt{C}_{\Psi}(\mathcal{I}, Y),\ \xi \in \mathtt{S}_w\right\}.$$

Let $\|\cdot\|_{\Upsilon}$ in $\mathtt{S}'_w$ be the seminorm defined as

$$\|\mathtt{u}\|_{\Upsilon} = \|\mathtt{u}_0\|_{\Upsilon} + \sup\{\|\mathtt{u}(\tau)\| : \tau \in [0, b]\},\ \mathtt{u} \in \mathtt{S}'_w.$$

Lemma 1 ([16])**.** *If $\mathtt{u} \in \mathtt{S}'_w$, then for $\varrho \in \mathcal{I}, \mathtt{u}_\varrho \in \mathtt{S}_w$. Moreover,*

$$\Upsilon|\mathtt{u}(\varrho)| \le \|\mathtt{u}_\varrho\|_{\Upsilon} \le \|\mathtt{u}_0\|_{\Upsilon} + \Upsilon \sup_{r\in[0,\varrho]} |\mathtt{u}(r)|, \quad \Upsilon = \int_{-\infty}^{0} w(\varrho)d\varrho < \infty.$$

Lemma 2 ([9])**.** *Let the linear operator, $\mathtt{A}$, be the infinitesimal generator of a C_0 semigroup iff:*

(c_i) *$\mathtt{A}$ is closed and $D(\mathtt{A}) = Y$;*
(c_{ii}) *$\rho(\mathtt{A})$ is the resolvent set of $\mathtt{A}$ containing $\mathbb{R}^+$ and $\forall\ \lambda > 0$, we write*

$$\|R(\lambda, \mathtt{A})\| \le \frac{1}{\lambda},$$

where $R(\lambda, \mathtt{A}) = (\lambda^\eta I - \mathtt{A})^{-1}z = \int_0^\infty e^{-\lambda^\alpha \varrho}\mathtt{T}(\varrho)zd\varrho$.

Definition 7. *The Wright-type function is defined as*

$$W_\eta(\varrho) = \sum_{k=0}^{\infty} \frac{(-z)^k}{k!\Gamma(-\eta k + 1 - \eta)}, \quad z \in \mathbb{C}$$

Proposition 1. *The Wright-type function, W_η, is an entire function that satisfies the following conditions:*

1. $W_\eta(\theta) \geq 0$ *for* $\theta \geq 0$, $\int_0^\infty W_\eta(\theta)d\theta = 1$;
2. $\int_0^\infty W_\eta(\theta)\theta^k d\theta = \frac{\Gamma(1+k)}{\Gamma(1+\eta k)}$, *for* $k > -1$;
3. $\int_0^\infty W_\eta(\theta)e^{z\theta}d\theta = E_\eta(-z)$, $z \in \mathbb{C}$.

Lemma 3. *The* Ψ*-HFD*$_{tial}$ *system (1) is equivalent to the integral equation*

$$\mathtt{u}(\varrho) = \frac{\left(\Psi(\rho) - \Psi(0)\right)^{(1-\eta)(\xi-1)}\left[\phi_0 - \mathtt{H}(0,\mathtt{u}(0))\right]}{\Gamma\left(\xi(1-\eta)+\xi\right)} + \mathtt{H}(\rho,\mathtt{u}_\rho) + \frac{1}{\Gamma(\eta)}\int_0^\varrho \left(\Psi(\varrho) - \Psi(\vartheta)\right)^{\eta-1}$$
$$\times \left[\mathtt{Au}(\varrho) + \mathtt{B}v(\varrho) + \mathtt{G}\left(\varrho, \mathtt{u}_\varrho, \int_0^\vartheta e(\varrho, s, \mathtt{u}_s)ds\right)\right]\Psi'(\vartheta)d\vartheta,$$

where $\varrho \in [0,b]$.

Proof. The proof is similar to the Lemma 3.1 in [28], so we omit it. □

For any $\mathtt{u} \in Y$, define the operators $\mathcal{S}_\Psi^{\eta,\xi}(\varrho,\vartheta)$, $\mathcal{Q}_\Psi^\eta(\varrho,\vartheta)$, and $\mathcal{P}_\Psi^\eta(\varrho,\vartheta)$ by

$$\mathcal{P}_\Psi^\eta(\varrho,\vartheta)\mathtt{u} = \int_0^\infty \zeta_\eta(\theta)\mathtt{T}\left((\Psi(\varrho) - \Psi(\vartheta))^\eta\theta\right)\mathtt{u}d\theta$$
$$\mathcal{S}_\Psi^{\eta,\xi}(\rho,\vartheta) = I_{0^+}^{(1-\eta)(1-\xi);\Psi}\mathcal{P}(\rho,\vartheta)\mathtt{u}$$

and

$$\mathcal{Q}_\Psi^\eta(\varrho,\vartheta)\mathtt{u} = \eta\int_0^\infty \theta\zeta_\eta(\theta)\mathtt{T}\left((\Psi(\varrho) - \Psi(\vartheta))^\eta\theta\right)\mathtt{u}d\theta,$$

for $0 \leq \vartheta \leq \varrho \leq b$ and the probability density function $\zeta_\eta(\theta) = \frac{1}{\eta}\theta^{-\frac{1}{\eta}-1}\rho_\eta(\theta^{-\frac{1}{\eta}})$ on $(0,\infty)$, i.e., $\zeta_\eta(\theta) \geq 0$ for $\theta \in (0,\infty)$ and $\int_0^\infty \zeta_\eta(\theta)d\theta = 1$.

Lemma 4 ([28])**.** *The operator* $\mathcal{S}_\Psi^{\eta,\xi}(\varrho,\vartheta)$ *and* $\mathcal{Q}_\Psi^\eta(\varrho,\vartheta)$ *hold the following properties:*

(a) *For any* $0 \leq \vartheta \leq \varrho$, $\mathcal{S}_\Psi^{\eta,\xi}(\varrho,\vartheta)$ *and* $\mathcal{Q}_\Psi^\eta(\varrho,\vartheta)$ *are bounded linear operators with*

$$\left\|\mathcal{S}_\Psi^{\eta,\xi}(\varrho,\vartheta)\mathtt{u}\right\| \leq \mathsf{L}'\|\mathtt{u}\| \quad \text{and} \quad \left\|\mathcal{Q}_\Psi^\eta(\varrho,\vartheta)\mathtt{u}\right\| \leq \mathsf{L}''\|\mathtt{u}\|$$

where $\mathsf{L}' = \frac{\kappa_\eta\left(\Psi(b)-\Psi(0)\right)^{(1-\eta)(1-\xi)}}{\Gamma(\eta+\xi-\eta\xi)}$ *and* $\mathsf{L}'' = \frac{\eta\kappa_\eta}{\Gamma(1+\eta)}$ *for all* $\mathtt{u} \in Y$.

(b) *The operators,* $\mathcal{S}_\Psi^{\eta,\xi}(\varrho,\vartheta)$ *and* $\mathcal{Q}_\Psi^\eta(\varrho,\vartheta)$*, are strongly continuous for all* $0 \leq \varrho_1 \leq \varrho_2 \leq b$*; thus, we write*

$$\left\|\mathcal{S}_\Psi^{\eta,\xi}(\varrho_2,\vartheta)\mathtt{u} - \mathcal{S}_\Psi^{\eta,\xi}(\varrho_2,\vartheta)\mathtt{u}\right\| \to 0 \quad \text{and} \quad \left\|\mathcal{Q}_\Psi^\eta(\varrho_2,\vartheta)\mathtt{u} - \mathcal{Q}_\Psi^\eta(\varrho_1,\vartheta)\mathtt{u}\right\| \to 0, \text{ as } \varrho_2 \to \varrho_1.$$

(c) *If* $\mathtt{T}(\varrho)$ *is a compact operator* $\forall\ \varrho > 0$, *then* $\mathcal{S}_\Psi^\eta(\varrho,\vartheta)$ *and* $\mathcal{Q}_\Psi^\eta(\varrho,\vartheta)$ *are compact for all* $\varrho,\ \vartheta > 0$.

(d) *If* $\mathcal{S}_\Psi^\eta(\varrho,\vartheta)$ *and* $\mathcal{Q}_\Psi^\eta(\varrho,\vartheta)$ *are the compact strongly continuous semigroup of bounded linear operators for* $\varrho,\ \vartheta > 0$, *then* $\mathcal{S}_\Psi^\eta(\varrho,\vartheta)$ *and* $\mathcal{Q}_\Psi^\eta(\varrho,\vartheta)$ *are continuous in the uniform operator topology.*

Lemma 5. *For any* $\mathtt{u} \in Y,\ \mu, \eta \in (0,1]$, *we have*

$$\mathtt{A}\mathcal{Q}_{\Psi}^{\eta}(\rho,\vartheta)\mathtt{u} = \mathtt{A}^{1-\mu}\mathcal{Q}_{\Psi}^{\eta}(\rho,\vartheta)\mathtt{A}^{\mu}\mathtt{u},\ \rho \in \mathcal{I};$$
$$\left\|\mathtt{A}^{\mu}\mathcal{Q}_{\Psi}^{\eta}(\rho,\vartheta)\right\| \leq \frac{\eta C_{\mu}\Gamma(2-\mu)}{\rho^{\eta\mu}\Gamma(1+\eta(1-\mu))}.$$

Definition 8. *A function,* $\mathtt{u} \in C([0,b],Y)$, *is called a mild solution of (1) if it satisfies*

$$\begin{aligned} \mathtt{u}(\varrho) = {} & \mathcal{S}_{\Psi}^{\eta,\xi}(\varrho,0)\left[\phi_0 - \mathtt{H}(0,\mathtt{u}(0))\right] + \mathtt{H}(\rho,\mathtt{u}_{\rho}) + \int_0^{\varrho}\left(\Psi(\varrho)-\Psi(\vartheta)\right)^{\eta-1}\mathtt{A}\mathcal{Q}_{\Psi}^{\eta}(\varrho,\vartheta)\mathtt{H}(\vartheta,\mathtt{u}_{\vartheta})\Psi'(\vartheta)d\vartheta \\ & + \int_0^{\varrho}\left(\Psi(\varrho)-\Psi(\vartheta)\right)^{\eta-1}\mathcal{Q}_{\Psi}^{\eta}(\varrho,\vartheta)\mathtt{B}v(\vartheta)\Psi'(\vartheta)d\vartheta \\ & + \int_0^{\varrho}\left(\Psi(\varrho)-\Psi(\vartheta)\right)^{\eta-1}\mathcal{Q}_{\Psi}^{\eta}(\varrho,\vartheta)\mathtt{G}\left(\varrho,\mathtt{u}_{\varrho},\int_0^{\vartheta}e(\varrho,s,\mathtt{u}_s)ds\right)\Psi'(\vartheta)d\vartheta, \quad for\ \varrho \in [0,b] \end{aligned} \tag{7}$$

Lemma 6 (Krasnoselskii's fixed point theorem [32])**.** *Let Y be a Banach space. Let $\mathfrak{D}$ be a bounded, closed, and convex subset of Y, and let P, Q be maps of $\mathfrak{D}$ into Y such that $Px + Qy \in \mathfrak{D}$, for each pair's $x, y \in \mathfrak{D}$. If P is contraction and Q is compact and continuous, then the equation $Px + Qx = x$ has a solution for $\mathfrak{D}$.*

We outline a suitable system, its operators, and its underlying presumptions as follows:

$$\begin{aligned} & {}^{H}D_{0^+}^{\eta,\xi;\Psi}\mathtt{u}(\varrho) = \mathtt{A}\mathtt{u}(\varrho) + \mathtt{B}v(\varrho),\ \varrho \in \mathcal{I}' = (0,b], \\ & I_{0^+}^{(1-\eta)(1-\xi);\Psi}\mathtt{u}(0) = \phi_0, \end{aligned} \tag{8}$$

and also define the following:

$$\mathfrak{T}_0^b = \int_0^b \left(\Psi(b)-\Psi(0)\right)^{\eta-1}\mathcal{Q}_{\Psi}^{\eta}(b,\delta)\mathtt{B}\mathtt{B}^*\mathcal{Q}_{\Psi}^{\eta^*}(b,\delta)d\delta,$$
$$R(\gamma,\mathfrak{T}_0^b) = \left(\gamma I + \mathfrak{T}_0^b\right)^{-1},\ \gamma > 0,$$

where $\mathtt{B}^*$ and $\mathcal{Q}_{\Psi}^{\eta^*}$ are the adjoint of $\mathtt{B}$ and $\mathcal{Q}_{\Psi}^{\eta}$, respectively, and $\mathfrak{T}_0^b$ is the linear bounded operator.

Then, $\forall\ \gamma > 0$ and $\mathtt{u}_1 \in Y$ take

$$v(\varrho) = \mathtt{B}^*\mathcal{Q}_{\Psi}^{\eta^*}(b,\varrho)\mathtt{R}(\gamma,\mathfrak{T}_0^b)P(\mathtt{u}(\cdot)),$$

where

$$\begin{aligned} P(q(\cdot)) = \mathtt{u}_1 - \Bigg[& \mathcal{S}_{\Psi}^{\eta,\xi}(\varrho,0)\left[\phi_0 - \mathtt{H}(0,\mathtt{u}(0))\right] - \mathtt{H}(b,\mathtt{u}_b) \\ & - \int_0^b \left(\Psi(b)-\Psi(\delta)\right)^{\eta-1}\mathtt{A}\mathcal{Q}_{\Psi}^{\eta}(b,\delta)\mathtt{H}(\delta,\mathtt{u}_{\delta})\Psi'(\delta)d\delta \\ & - \int_0^b \left(\Psi(b)-\Psi(\delta)\right)^{\eta-1}\mathcal{Q}_{\Psi}^{\eta}(b,\delta)\mathtt{G}\left(\omega,\mathtt{u}_{\omega},\int_0^{\omega}e(\omega,s,\mathtt{u}_s)ds\right)\Psi'(\delta)d\delta\Bigg]. \end{aligned}$$

Consider the following hypotheses:

(H_1) $\{\mathtt{T}(\varrho)\}_{t\geq 0}$ is the C_0-semigroup , such that $\sup_{\varrho\in[0,\infty)}\|\mathtt{T}(\varrho)\| = M_{\eta}$, *where* $M_{\eta} \geq 1$.

(H_2) For $\varrho \in \mathcal{I}$, $\mathtt{G}(\varrho,\cdot,\cdot) : \mathtt{S}_w \times Y \to Y$, $e(\varrho,s,\cdot) : \mathtt{S}_w \to Y$ are continuous functions, and for each $\mathtt{u} \in \mathcal{X}$, $\mathtt{G}(\cdot,\mathtt{u}_{\varrho},\int e(\rho,s,\mathtt{u}_s)) : \mathcal{I} \to Y$ and $e(\cdot,\cdot,\mathtt{u}_{\varrho}) : \mathcal{I}\times\mathcal{I} \to Y$ are strongly measurable.

(H_3) There exists an increasing function $\Lambda : \mathbb{R}^+ \to (0, \infty)$ and $L_{\mathtt{G},\mathtt{r}}(\cdot) \in L^1(\mathcal{I}', \mathbb{R})$, such that $\|\mathtt{G}(\varrho, \gamma_1, \gamma_2)\| \leq L_{\mathtt{G},\mathtt{r}}(\varrho)\Lambda(\|\gamma_1\|_{\mathtt{Y}} + \|\gamma_2\|)$ for all $(\varrho, \gamma_1, \gamma_2) \in \mathcal{I} \times \mathtt{S}_w \times Y$, and $\exists$ a constant $\mathsf{M} > 0$, then

$$\lim_{\mathtt{r}\to\infty} \frac{L_{\mathtt{G},\mathtt{r}}(\varrho)\Lambda(\|\gamma_1\|_{\mathtt{Y}} + \|\gamma_2\|)}{\mathtt{r}} = \mathsf{M}_1$$

(H_4) There exists a constant $E_0 > 0$, such that: $\|e(\varrho, s, \gamma)\| \leq E_0(1 + \|\gamma\|_{\mathtt{Y}})$ for all $(\varrho, s, \gamma) \in \mathcal{I} \times \mathcal{I} \times \mathtt{S}_w$.

(H_5) The function $\mathtt{H} : \mathcal{I} \times \mathtt{S}_w \to Y$ is continuous, and there exists $0 < \mu < 1$, $\mathtt{H} \in D(\mathtt{A}^\mu)$ for any $\mathtt{u} \in Y$, $\mathtt{A}^\mu\mathtt{H}(\cdot, \mathtt{u})$ is strongly measurable, there exists $\mathtt{K}_\mathtt{H}$, $\mathtt{K}'_\mathtt{H} > 0$ such that:

$$\|\mathtt{A}^\mu\mathtt{H}(\rho, l_1(\rho)) - \mathtt{A}^\mu\mathtt{H}(\rho, l_2(\rho))\| \leq \mathtt{K}_\mathtt{H}(\Psi(\rho) - \Psi(0))^{(1-\eta)(1-\xi)}\|l_1(\rho) - l_2(\rho)\|_{\mathtt{Y}},$$
$$\|\mathtt{A}^\mu\mathtt{H}(\rho, \mathtt{u}(\rho))\| \leq \mathtt{K}'_\mathtt{H}\left(1 + (\Psi(\rho) - \Psi(0))^{(1-\eta)(1-\xi)}\|\mathtt{u}\|_{\mathtt{Y}}\right),$$

and there exists a constant M_2 such that:

$$\lim_{\mathtt{r}\to 0} \frac{\mathtt{K}'_\mathtt{H}(1 + \mathtt{r}')}{\mathtt{r}} = \mathsf{M}_2$$

3. Approximate Controllability

Theorem 1. *Assume (H_1)–(H_5) satisfy. Then, Equation (1) has at least a mild solution for $\mathcal{I}$ with:*

$$(1 - \mathsf{L}''\mathtt{K}_\mathtt{B}^2)\left[\mathsf{L}'\mathsf{M}_2 + \frac{\eta C_{(1-\mu)}\Gamma(1-\mu)}{b^{\eta(1-\mu)}\Gamma(1+\eta\mu)}\mathsf{M}_2 + \mathsf{L}''\mathsf{M}_1\right] \leq 1,$$

Proof. Consider the operator $\Phi : \mathtt{S}'_w \to \mathtt{S}'_w$, defined by

$$\Phi(\mathtt{u}(\varrho)) = \begin{cases} \Phi_1(\varrho), \quad (-\infty, 0], \\ \mathcal{S}_\Psi^{\eta,\zeta}(\varrho, 0)[\phi_0 + \mathtt{H}(0, \mathtt{u}(0))] + \mathtt{H}(\rho, \mathtt{u}_\vartheta) \\ + \int_0^\varrho (\Psi(\rho) - \Psi(\vartheta))^{\eta-1}\mathtt{A}\mathcal{Q}_\Psi^\eta(\rho, \vartheta)\mathtt{H}(\rho, \mathtt{u}_\vartheta)\Psi'(\vartheta)d\vartheta \\ + \int_0^\varrho (\Psi(\varrho) - \Psi(\vartheta))^{\eta-1}\mathcal{Q}_\Psi^\eta(\varrho, \vartheta)\Psi'(\varrho)d\vartheta\mathtt{G}\left(\vartheta, \mathtt{u}_\vartheta, \int_0^\vartheta e(\vartheta, s, \mathtt{u}_s)ds\right)d\vartheta \\ + \int_0^\varrho (\Psi(\varrho) - \Psi(\vartheta))^{\eta-1}\mathcal{Q}_\Psi^\eta(\varrho, \vartheta)\mathtt{B}v(\varrho)\Psi'(\vartheta)d\vartheta, \quad \varrho \in (0, b] \end{cases} \tag{9}$$

For $\Phi_1 \in \mathtt{S}_w$, we define $\widehat{\Phi}$ by

$$\widehat{\Phi}(\varrho) = \begin{cases} \Phi_1(\varrho) \quad \varrho \in (-\infty, 0], \\ \mathcal{S}_\Psi^{\eta,\zeta}(\varrho, 0)\phi_0, \quad \varrho \in \mathcal{I}, \end{cases}$$

Then, $\widehat{\Phi} \in \mathtt{S}'_w$. Let $\mathtt{u}_\varrho = [y_\varrho + \widehat{\Phi}_\varrho]$, $\infty < \varrho \leq b$. It can be easily shown that $\mathtt{u}$ satisfies from (8) iff v satisfies y_0 and

$$y(\varrho) = \mathcal{S}_{\Psi}^{\eta,\xi}(\rho,0)\mathtt{H}(0,\mathtt{u}(0)) + \mathtt{H}(\rho, y_\varrho + \widehat{\Phi}_\varrho) + \int_0^\varrho (\Psi(\rho)-\Psi(\vartheta))^{\eta-1}\mathtt{A}\mathcal{Q}_{\Psi}^{\eta}(\rho,\vartheta)\mathtt{H}(\rho, y_\vartheta + \widehat{\Phi}_\vartheta)\Psi'(\vartheta)d\vartheta$$
$$+ \int_0^\varrho (\Psi(\varrho)-\Psi(\vartheta))^{\eta-1}\mathcal{Q}_{\Psi}^{\eta}(\varrho,\vartheta)\mathtt{G}\left(\vartheta,(y_\vartheta+\widehat{\Phi}_\vartheta),\int_0^\vartheta e(\vartheta,s,y_s+\widehat{\Phi}_s)ds\right)\Psi'(\vartheta)d\vartheta$$
$$+ \int_0^\varrho (\Psi(\varrho)-\Psi(\vartheta))^{\eta-1}\mathcal{Q}_{\Psi}^{\eta}(\varrho,\vartheta)\mathtt{B}\mathtt{B}^*\mathcal{Q}_\eta^*(b,\vartheta)R(\alpha,\mathfrak{T}_0^b)\Big[\mathtt{u}_1 - \mathcal{S}_{\Psi}^{\eta,\xi}(\varrho,0)[\phi_0 - \mathtt{H}(0,\mathtt{u}(0))]$$
$$- \mathtt{H}(\rho, y_\varrho+\widehat{\Phi}_\varrho) - \int_0^b (\Psi(b)-\Psi(\delta))^{\eta-1}\mathtt{A}\mathcal{Q}_{\Psi}^{\eta}(b,\delta)\mathtt{H}(\delta,y_\delta+\widehat{\Phi}_\delta)\Psi'(\delta)d\delta$$
$$- \int_0^b (\Psi(b)-\Psi(\delta))^{\eta-1}\mathcal{Q}_{\Psi}^{\eta}(b,\delta)\mathtt{G}\left(\delta, y_\delta+\widehat{\Phi}_\delta, \int_0^\delta e(\delta,s,y_s+\widehat{\Phi}_s)ds\right)\Psi'(\delta)d\delta\Big]\Psi'(\vartheta)d\vartheta.$$

Let $\mathtt{S}''_w = \{y \in \mathtt{S}'_w : y_0 \in \mathtt{S}_w\}$. For any $y \in \mathtt{S}'_w$,

$$\begin{aligned}\|y\|_{\mathrm{Y}} &= \|y_0\|_{\mathrm{Y}} + \sup\{\|y(\omega)\| : 0 \le \omega \le b\} \\ &= \sup\{\|y(\omega)\| : 0 \le \omega \le b\}.\end{aligned}$$

Thus, $(\mathtt{S}''_w, \|\cdot\|_{\mathrm{Y}})$ is a Banach space.

For $\mathtt{r} > 0$, choose $\mathtt{S}_\mathtt{r} = \{y \in \mathtt{S}''_w : \|y\|_{\mathrm{Y}} \le \mathtt{r}\}$; then, $\mathtt{S}_\mathtt{r} \subset \mathtt{S}''_w$ is uniformly bounded, and for $y \in \mathtt{S}_\mathtt{r}$, by Lemma 1,

$$\begin{aligned}\|y_\varrho + \widehat{\Phi}_\varrho\|_{\mathrm{Y}} &\le \|y_\varrho\|_{\mathrm{Y}} + \|\widehat{\Phi}_\varrho\|_{\mathrm{Y}} \\ &\le \mathrm{Y}(\mathtt{r} + \mathsf{L}'\phi_0) + \|\Phi_1\|_{\mathrm{Y}} \\ &= \mathtt{r}'\end{aligned} \tag{10}$$

Consider the operator $\Phi : \mathtt{S}''_w \to \mathtt{S}''_w$, defined by

$$\Phi' y(\varrho) = \begin{cases} 0, \ \varrho \in (-\infty, 0], \\ \mathcal{S}_{\Psi}^{\eta,\xi}(\rho,0)\mathtt{H}(0,\mathtt{u}(0)) + \mathtt{H}(\rho, y_\rho + \widehat{\Phi}_\rho) + \int_0^\varrho (\Psi(\rho)-\Psi(\vartheta))^{\eta-1} \\ \times \mathtt{A}\mathcal{Q}_{\Psi}^{\eta}(\rho,\vartheta)\mathtt{H}(\vartheta, y_\vartheta+\widehat{\Phi}_\vartheta)\Psi'(\vartheta)d\vartheta \\ + \int_0^\varrho (\Psi(\varrho)-\Psi(\vartheta))^{\eta-1}\mathcal{Q}_{\Psi}^{\eta}(\varrho,\vartheta)\mathtt{G}\left(\vartheta, y_\vartheta+\widehat{\Phi}_\vartheta, \int_0^\vartheta e(\vartheta,s,y_s+\widehat{\Phi}_s)ds\right)\Psi'(\vartheta)d\vartheta \\ + \int_0^\varrho (\Psi(\varrho)-\Psi(\vartheta))^{\eta-1}\mathcal{Q}_{\Psi}^{\eta}(\varrho,\vartheta)\mathtt{B}v(\varrho)\Psi'(\vartheta)d\vartheta, \quad \varrho \in \mathcal{I}. \end{cases}$$

Here, we prove Φ has a fixed point. Then, for $\rho \in \mathcal{I}$, the operator Φ' can be decomposed:
$\Phi' = \Phi'_1 + \Psi'_2$, where

$$\Phi'_1 = \mathcal{S}_{\Psi}^{\eta,\xi}(\rho,0)\mathtt{H}(0,\mathtt{u}(0)) + \mathtt{H}(\rho, y_\rho+\widehat{\Phi}_\rho) + \int_0^\varrho (\Psi(\rho)-\Psi(\vartheta))^{\eta-1}\mathtt{A}\mathcal{Q}_{\Psi}^{\eta}(\rho,\vartheta)\mathtt{H}(\vartheta, y_\vartheta+\widehat{\Phi}_\vartheta)\Psi'(\vartheta)d\vartheta,$$
$$\Phi'_2 = \int_0^\varrho (\Psi(\varrho)-\Psi(\vartheta))^{\eta-1}\mathcal{Q}_\eta(\varrho,\vartheta)\mathtt{G}\left(\vartheta, y_\vartheta+\widehat{\Phi}_\vartheta, \int_0^\vartheta e(\vartheta,s,y_s+\widehat{\Phi}_s)ds\right)\Psi'(\vartheta)d\vartheta$$
$$+ \int_0^\varrho (\Psi(\varrho)-\Psi(\vartheta))^{\eta-1}\mathcal{Q}_\eta(\varrho,\vartheta)\mathtt{B}v(\varrho)\Psi'(\vartheta)d\vartheta.$$

Step 1. We show that $\Phi'(y(\varrho)) \in \mathtt{S}_\mathtt{r}$ to prove $\Phi'(\mathtt{S}_\mathtt{r}) \subset \mathtt{S}_\mathtt{r}$. We assume that for each $\mathtt{r} > 0$, there exists $\varrho \in [0, b]$, such that

$$\|(\Phi' y)(\varrho)\| > \mathtt{r}. \tag{11}$$

Because

$$\begin{aligned}
\left\|\left(\Phi' y\right)(\varrho)\right\| \leq & \sup_{\rho\in[0,b]} \big(\Psi(\rho)-\Psi(0)\big)^{(1-\eta)(1-\xi)}\bigg[\left\|\mathcal{S}_{\Psi}^{\eta,\xi}(\rho,0)\mathtt{H}\big(0,\mathtt{u}(0)\big)\right\| + \left\|\mathtt{H}\big(\rho,y_\rho+\widehat{\Phi}_\rho\big)\right\| \\
& + \left\|\int_0^{\varrho}\big(\Psi(\rho)-\Psi(\vartheta)\big)^{\eta-1}\mathtt{A}\mathcal{Q}_{\Psi}^{\eta}(\rho,\vartheta)\mathtt{H}\big(\vartheta,y_\vartheta+\widehat{\Phi}_\vartheta\big)\Psi'(\vartheta)d\vartheta\right\| \\
& + \left\|\int_0^{\varrho}(\Psi(\varrho)-\Psi(\vartheta))^{\eta-1}\mathcal{Q}_{\eta}(\varrho,\vartheta)\mathtt{G}\bigg(\vartheta,y_\vartheta+\widehat{\Phi}_\vartheta,\int_0^{\vartheta}e\big(\vartheta,s,y_s+\widehat{\Phi}_s\big)ds\bigg)\Psi'(\vartheta)d\vartheta\right\| \\
& + \left\|\int_0^{\varrho}(\Psi(\varrho)-\Psi(\vartheta))^{\eta-1}\mathcal{Q}_{\eta}(\varrho,\vartheta)\mathtt{B}v(\rho)\Psi'(\vartheta)d\vartheta\right\|\bigg] \\
= & \sum_{j=1}^{5} I_j,
\end{aligned}$$

where

$$\begin{aligned}
I_1 = & \sup_{\rho\in[0,b]} \big(\Psi(\rho)-\Psi(0)\big)^{(1-\eta)(1-\xi)}\left\|\mathcal{S}_{\Psi}^{\eta,\xi}(\rho,0)\mathtt{H}\big(0,\mathtt{u}(0)\big)\right\| \\
\leq & \big(\Psi(b)-\Psi(0)\big)^{(1-\eta)(1-\xi)}\mathsf{L}'\mathtt{K}_{\mathtt{H}}'\mathtt{K}^*\|\phi_0\|, \\
I_2 = & \sup_{\rho\in[0,b]} \big(\Psi(\rho)-\Psi(0)\big)^{(1-\eta)(1-\xi)}\left\|\mathtt{H}\big(\rho,y_\rho+\widehat{\Phi}_\rho\big)\right\| \\
\leq & \big(\Psi(b)-\Psi(0)\big)^{(1-\eta)(1-\xi)}\mathsf{L}'\mathtt{K}_{\mathtt{H}}'\big[1+\mathtt{r}'\big], \\
I_3 = & \sup_{\rho\in[0,b]} \big(\Psi(\rho)-\Psi(0)\big)^{(1-\eta)(1-\xi)}\left\|\int_0^{\varrho}\big(\Psi(\rho)-\Psi(\vartheta)\big)^{\eta-1}\mathtt{A}\mathcal{Q}_{\Psi}^{\eta}(\rho,\vartheta)\mathtt{H}\big(\vartheta,y_\vartheta+\widehat{\Phi}_\vartheta\big)\Psi'(\vartheta)d\vartheta\right\| \\
\leq & \big(\Psi(b)-\Psi(0)\big)^{(1-\eta)(1-\xi)}\frac{\eta C_{(1-\mu)}\mathtt{K}_{\mathtt{H}}'\big[1+\mathtt{r}'\big]\Gamma(1-\mu)}{b^{\eta(1-\mu)}\Gamma(1+\eta\mu)} \\
& \times\int_0^{\rho}\big(\Psi(\rho)-\Psi(\vartheta)\big)^{\eta-1}\Psi'(\vartheta)d\vartheta \\
\leq & \frac{\eta C_{(1-\mu)}\mathtt{K}_{\mathtt{H}}'\big[1+\mathtt{r}'\big]\Gamma(1-\mu)}{b^{\eta(1-\mu)}\Gamma(1+\eta\mu)}\big(\Psi(b)-\Psi(0)\big)^{1-\xi+\eta\xi} \\
I_4 = & \sup_{\rho\in[0,b]} \big(\Psi(\rho)-\Psi(0)\big)^{(1-\eta)(1-\xi)}\bigg\|\int_0^{\varrho}(\Psi(\varrho)-\Psi(\vartheta))^{\eta-1}\mathcal{Q}_{\eta}(\varrho,\vartheta) \\
& \times\mathtt{G}\bigg(\vartheta,y_\vartheta+\widehat{\Phi}_\vartheta,\int_0^{\vartheta}e\big(\vartheta,s,y_s+\widehat{\Phi}_s\big)ds\bigg)\Psi'(\vartheta)d\vartheta\bigg\| \\
\leq & \big(\Psi(b)-\Psi(0)\big)^{(1-\eta)(1-\xi)}\mathsf{L}''L_{\mathtt{G},\mathtt{r}}(b)\Lambda\big(\mathtt{r}'+E_0(1+\mathtt{r}')\big)\int_0^{\varrho}\big(\Psi(\varrho)-\Psi(\vartheta)\big)^{\eta-1}\Psi'(\vartheta)d\vartheta \\
\leq & \mathsf{L}''L_{\mathtt{G},\mathtt{r}}(b)\Lambda\big(\mathtt{r}'+E_0(1+\mathtt{r}')\big)\big(\Psi(b)-\Psi(0)\big)^{1-\xi+\eta\xi} \\
I_5 = & \sup_{\rho\in[0,b]} \big(\Psi(\rho)-\Psi(0)\big)^{(1-\eta)(1-\xi)}\left\|\int_0^{\varrho}(\Psi(\varrho)-\Psi(\vartheta))^{\eta-1}\mathcal{Q}_{\eta}(\varrho,\vartheta)\mathtt{B}v(\rho)\Psi'(\vartheta)d\vartheta\right\| \\
\leq & \big(\Psi(b)-\Psi(0)\big)^{1-\xi+\eta\xi}\mathsf{L}''^2\mathtt{K}_{\mathtt{B}}^2\frac{1}{\alpha}\big[\mathtt{u}_1-\mathsf{L}'\phi_0\big]+\mathsf{L}''^2\mathtt{K}_{\mathtt{B}}^2\frac{1}{\alpha}\big[I_1-I_2-I_3-I_4\big].
\end{aligned}$$

Thus, we obtain the sum, dividing both sides by $\mathtt{r}$ and applying the limit as $\mathtt{r}\to\infty$,

$$1 < \big(1-\mathsf{L}''\mathtt{K}_{\mathtt{B}}^2\big)\bigg[\mathsf{L}'\mathsf{M}_2+\frac{\eta C_{(1-\mu)}\Gamma(1-\mu)}{b^{\eta(1-\mu)}\Gamma(1+\eta\mu)}\mathsf{M}_2+\mathsf{L}''\mathsf{M}_1\bigg],$$

Then, we obtain a contradiction to our assumption.

Step 2. To prove that Φ_1' is contraction, let $y_1, y_2 \in \mathtt{S}_{\mathtt{r}}$, abd we obtain

$$\begin{aligned}&\left\|\Phi_1'(y_1(\rho))-\Phi_1'(y_2(\rho))\right\|\\&\le \sup\left(\Psi(\rho)-\Psi(0)\right)^{(1-\eta)(1-\xi)}\left[\left\|\mathtt{H}\left(\rho,y_{1\rho}+\widehat{\Phi}_\rho\right)-\mathtt{H}\left(\rho,y_{2\rho}+\widehat{\Phi}_\rho\right)\right\|\right.\\&\left.+\int_0^{\varrho}\left(\Psi(\rho)-\Psi(\vartheta)\right)^{\eta-1}\left\|\mathtt{A}\mathcal{Q}_\Psi^\eta(\rho,\vartheta)\right\|\left\|\mathtt{H}\left(\vartheta,y_{1\vartheta}+\widehat{\Phi}_\vartheta\right)-\mathtt{H}\left(\vartheta,y_{2\vartheta}+\widehat{\Phi}_\vartheta\right)\right\|\Psi'(\vartheta)d\vartheta\right]\\&\le\left(\Psi(b)-\Psi(0)\right)^{(1-\eta)(1-\xi)}\left[\left(\left\|\mathtt{A}^{-\mu}\right\|+\int_0^{\varrho}\left(\Psi(\rho)-\Psi(\vartheta)\right)^{\eta-1}\left\|\mathtt{A}^{1-\mu}\mathcal{Q}_\Psi^\eta(\rho,\vartheta)\right\|\right)\right.\\&\left.\times\left\|\mathtt{A}^\mu\mathtt{H}\left(\vartheta,y_{1\vartheta}+\widehat{\Phi}_\vartheta\right)-\mathtt{A}^\mu\mathtt{H}\left(\vartheta,y_{2\vartheta}+\widehat{\Phi}_\vartheta\right)\right\|\Psi'(\vartheta)d\vartheta\right],\end{aligned}$$

From the hypotheses (H_5), we obtain

$$\left\|\mathtt{A}^\mu\mathtt{H}\left(\rho,y_{1\rho}+\widehat{\Phi}_\rho\right)-\mathtt{A}^\mu\mathtt{H}\left(\rho,y_{2\rho}+\widehat{\Phi}_\rho\right)\right\|\le\mathtt{K}_\mathtt{H}\left(\Psi(\rho)-\Psi(0)\right)^{(1-\eta)(1-\xi)}\left\|y_{1\rho}-y_{2\rho}\right\|_{\mathrm{Y}}. \quad (12)$$

Using Lemmas 12 and 5,

$$\left\|\Phi_1'(y_1(\rho))-\Phi_1'(y_2(\rho))\right\|\le\mathtt{L}^*\left(\Psi(\rho)-\Psi(0)\right)^{(1-\eta)(1-\xi)}\left\|y_{1\rho}-y_{2\rho}\right\|_{\mathrm{Y}}.$$

Therefore, Φ_1' is a contraction.

Step 3. To prove Φ_2' is completely continuous, first, we have to prove Φ_2' is continuous. Let

$$\begin{aligned}\Phi_2'=&\int_0^{\varrho}(\Psi(\varrho)-\Psi(\vartheta))^{\eta-1}\mathcal{Q}_\eta(\varrho,\vartheta)\mathtt{G}\left(\vartheta,y_\vartheta+\widehat{\Phi}_\vartheta,\int_0^{\vartheta}e\left(\vartheta,s,y_s+\widehat{\Phi}_s\right)ds\right)\Psi'(\vartheta)d\vartheta\\&+\int_0^{\varrho}(\Psi(\varrho)-\Psi(\vartheta))^{\eta-1}\mathcal{Q}_\eta(\varrho,\vartheta)\mathtt{B}v(\varrho)\Psi'(\vartheta)d\vartheta.\end{aligned}$$

Take $\{y^k\}\subset\mathtt{S_r}$ such that $y^k\to y\in\mathtt{S_r}$ as $k\to\infty$. From hypotheses (H_2) and (H_3), we can write, for each $\varrho\in\mathcal{I}$,

$$\mathtt{G}\left(\varrho,y_\varrho^k+\widehat{\Phi}_\varrho,\int_0^{\varrho}e\left(\varrho,s,y_s^k+\widehat{\Phi}_s\right)\right)\to\mathtt{G}\left(\varrho,y_\varrho+\widehat{\Phi}_\varrho,\int_0^{\varrho}e\left(\varrho,s,y_s+\widehat{\Phi}_s\right)\right)\ as\ k\to\infty\quad for\ all\ k\in\mathbb{N}. \quad (13)$$

From hypotheses (H_5)

$$\mathtt{H}\left(\delta,y_\delta^k+\widehat{\Phi}_\delta\right)\to\mathtt{H}\left(\delta,y_\delta+\widehat{\Phi}_\delta\right)\ as\ k\to\infty\ for\ all\ k\in\mathbb{N}. \quad (14)$$

Using Lebesgue dominated convergence theorem, for any $\varrho\in\mathcal{I}$, we write

$$
\begin{aligned}
&\left\|\left(\Phi_2' y^k\right)(\varrho)-\left(\Phi_2 y\right)(\varrho)\right\| \\
&\leq \left\| \sup\left(\Psi(\rho)-\Psi(0)\right)^{(1-\eta)(1-\xi)} \int_0^{\varrho}\left(\Psi(\varrho)-\Psi(\vartheta)\right)^{\eta-1} \mathcal{Q}_{\Psi}^{\eta}(\varrho, \vartheta) \Psi'(\vartheta)\right. \\
&\quad \times \left[\mathtt{G}\left(\vartheta, y_{\vartheta}^k+\widehat{\Phi}_{\vartheta}, \int_0^{\vartheta} e\left(\vartheta, s, y_s^k+\widehat{\Phi}_s\right) d\varrho\right)-\mathtt{G}\left(\vartheta, y_{\vartheta}+\widehat{\Phi}_{\vartheta}, \int_0^{\vartheta} e\left(\vartheta, s, y_s+\widehat{\Phi}_s\right) d\varrho\right)\right] d\vartheta \Bigg\| \\
&\quad - \Bigg\| \sup\left(\Psi(\rho)-\Psi(0)\right)^{(1-\eta)(1-\xi)} \int_0^{\varrho}\left(\Psi(\varrho)-\Psi(\vartheta)\right)^{\eta-1} \mathcal{Q}_{\Psi}^{\eta}(\varrho, \vartheta) \Psi'(\vartheta) \mathtt{B}\mathtt{B}^* \mathcal{Q}_{\Psi}^{\eta *}(b, \varrho) R\left(\alpha, \mathfrak{T}_0^b\right) \\
&\quad \times \Bigg[\mathtt{H}\left(\delta, y_{\delta}^k+\widehat{\Phi}_{\delta}\right)-\mathtt{H}\left(\delta, y_{\delta}+\widehat{\Phi}_{\delta}\right) \\
&\quad + \int_0^b\left(\Psi(b)-\Psi(\delta)\right)^{\eta-1}\left[\mathtt{A} \mathcal{Q}_{\Psi}^{\eta}(b, \delta)\left[\mathtt{H}\left(\delta, y_{\delta}^k+\widehat{\Phi}_{\delta}\right)-\mathtt{H}\left(\delta, y_{\delta}+\widehat{\Phi}_{\delta}\right)\right]\right] \Psi'(\delta) d\delta \\
&\quad + \int_0^b\left(\Psi(b)-\Psi(\delta)\right)^{\eta-1}\Bigg(\mathtt{G}\left(\delta, y_{\delta}^k+\widehat{\Phi}_{\delta}, \int_0^{\delta} e\left(\delta, s, y_s^k+\widehat{\Phi}_s\right) d\delta\right) \\
&\quad - \mathtt{G}\left(\delta, y_{\delta}+\widehat{\Phi}_{\delta}, \int_0^{\delta} e\left(\delta, s, y_s+\widehat{\Phi}_s\right) d\delta\right)\Bigg) \Psi'(\delta) d\delta\Bigg] d\vartheta \Bigg\| \\
&\leq \mathtt{L}'' \mathtt{K}_{\mathtt{B}}\left(\Psi(b)-\Psi(0)\right)^{(1-\eta)(1-\xi)} \int_0^{\varrho}\left(\Psi(\varrho)-\Psi(\vartheta)\right)^{\eta-1} \Psi'(\vartheta) \\
&\quad \times \left\|\mathtt{G}\left(\vartheta, y_{\vartheta}^k+\widehat{\Phi}_{\vartheta}, \int_0^{\vartheta} e\left(\vartheta, s, y_s^k+\widehat{\Phi}_s\right) d\varrho\right)-\mathtt{G}\left(\vartheta, y_{\vartheta}+\widehat{\Phi}_{\vartheta}, \int_0^{\vartheta} e\left(\vartheta, s, y_s+\widehat{\Phi}_s\right) d\varrho\right)\right\| d\vartheta \\
&\quad - \frac{\left(\mathtt{L}'' \mathtt{K}_{\mathtt{B}}\right)^2}{\alpha}\left(\Psi(b)-\Psi(0)\right)^{(1-\eta)(1-\xi)} \int_0^{\varrho}\left(\Psi(\varrho)-\Psi(\vartheta)\right)^{\eta-1}\left[\left\|\mathtt{H}\left(\delta, y_{\delta}^k+\widehat{\Phi}_{\delta}\right)-\mathtt{H}\left(\delta, y_{\delta}+\widehat{\Phi}_{\delta}\right)\right\|\right. \\
&\quad + \int_0^b\left(\Psi(b)-\Psi(\delta)\right)^{\eta-1}\left\|\mathtt{A} \mathcal{Q}_{\Psi}^{\eta}(b, \delta)\right\|\left\|\mathtt{H}\left(\delta, y_{\delta}^k+\widehat{\Phi}_{\delta}\right)-\mathtt{H}\left(\delta, y_{\delta}+\widehat{\Phi}_{\delta}\right)\right\| \Psi'(\delta) d\delta \\
&\quad + \int_0^b\left(\Psi(b)-\Psi(\delta)\right)^{\eta-1}\Bigg\|\mathtt{G}\left(\delta, y_{\delta}^k+\widehat{\Phi}_{\delta}, \int_0^{\delta} e\left(\delta, s, y_s^k+\widehat{\Phi}_s\right) d\delta\right) \\
&\quad \left.- \mathtt{G}\left(\delta, y_{\delta}+\widehat{\Phi}_{\delta}, \int_0^{\delta} e\left(\delta, s, y_s+\widehat{\Phi}_s\right) d\delta\right)\right\| \Psi'(\delta) d\delta\Bigg] d\vartheta .
\end{aligned}
$$

Apply $k \to \infty$ from (13) and (14) $\Longrightarrow \left\|\left(\Phi_2' y^k\right)(\varrho)-\left(\Phi_2' y\right)(\varrho)\right\| \to 0$. Hence, Φ is continuous.

Next, we show that $\left\{(\Phi_2' y)(\varrho) : y \in \mathtt{S}_{\mathtt{r}}\right\}$ is equicontinuous in Y. For any $y \in \mathtt{S}_{\mathtt{r}}$ and $0 \leq \varrho_1 \leq \varrho_2 \leq b$, we have

$$
\begin{aligned}
&\left\|(\Phi_2'y)(\varrho_2)-(\Phi_2'y)(\varrho_1)\right\| \\
&\leq \Bigg\|\sup\big(\Psi(\rho_2)-\Psi(0)\big)^{(1-\eta)(1-\xi)}\int_0^{\varrho_2}\big(\Psi(\varrho_2)-\Psi(\vartheta)\big)^{\eta-1}\mathcal{Q}_\Psi^\eta(\varrho_2,\vartheta) \\
&\quad\times \mathtt{G}\bigg(\vartheta,y_\vartheta+\widehat{\Phi}_\vartheta,\int_0^{\varrho}e\big(\vartheta,s,y_s+\widehat{\Phi}_s\big)\bigg)\Psi'(\vartheta)d\vartheta \\
&\quad-\sup\big(\Psi(\rho_1)-\Psi(0)\big)^{(1-\eta)(1-\xi)}\int_0^{\varrho_1}\big(\Psi(\varrho_1)-\Psi(\vartheta)\big)^{\eta-1}\mathcal{Q}_\Psi^\eta(\varrho_1,\vartheta) \\
&\quad\times \mathtt{G}\bigg(\vartheta,y_\vartheta+\widehat{\Phi}_\vartheta,\int_0^{\vartheta}e\big(\vartheta,s,y_s+\widehat{\Phi}_s\big)\bigg)\Psi'(\vartheta)d\vartheta\Bigg\| \\
&\quad+\Bigg\|\sup\big(\Psi(\rho_2)-\Psi(0)\big)^{(1-\eta)(1-\xi)}\int_0^{\varrho_2}\big(\Psi(\varrho_2)-\Psi(\vartheta)\big)^{\eta-1}\mathcal{Q}_\Psi^\eta(\varrho_2,\vartheta)\mathtt{B}v(\vartheta)\Psi'(\vartheta)d\vartheta \\
&\quad-\int_0^{\varrho_1}\big(\Psi(\varrho_1)-\Psi(\vartheta)\big)^{\eta-1}\mathcal{Q}_\Psi^\eta(\varrho_1,\vartheta)\mathtt{B}v(\vartheta)\Psi'(\vartheta)d\vartheta\Bigg\| \\
&\leq \big(\Psi(b)-\Psi(0)\big)^{(1-\eta)(1-\xi)}\Bigg[\Bigg\|\int_{\varrho_1}^{\varrho_2}\big(\Psi(\varrho_2)-\Psi(\vartheta)\big)^{\eta-1}\mathcal{Q}_\Psi^\eta(\varrho_2,\vartheta) \\
&\quad\times \mathtt{G}\bigg(\vartheta,y_\vartheta+\widehat{\Phi}_\vartheta,\int_0^{\varrho}e\big(\vartheta,s,y_s+\widehat{\Phi}_s\big)\bigg)d\vartheta\Bigg\| \\
&\quad+\Bigg\|\int_0^{\varrho_1}\Big[\big(\Psi(\varrho_2)-\Psi(\vartheta)\big)^{\eta-1}-\big(\Psi(\varrho_1)-\Psi(\vartheta)\big)^{\eta-1}\Big]\mathcal{Q}_\Psi^\eta(\varrho_2,\vartheta) \\
&\quad\times \mathtt{G}\bigg(\vartheta,y_\vartheta+\widehat{\Phi}_\vartheta,\int_0^{\varrho}e\big(\vartheta,s,y_s+\widehat{\Phi}_s\big)\bigg)\Psi'(\vartheta)d\vartheta\Bigg\| \\
&\quad+\Bigg\|\int_0^{\varrho_1}\big(\Psi(\varrho_1)-\Psi(\vartheta)\big)^{\eta-1}\Big[\mathcal{Q}_\Psi^\eta(\varrho_2,\vartheta)-\mathcal{Q}_\Psi^\eta(\varrho_1,\vartheta)\Big] \\
&\quad\times \mathtt{G}\bigg(\vartheta,y_\vartheta+\widehat{\Phi}_\vartheta,\int_0^{\varrho}e\big(\vartheta,s,y_s+\widehat{\Phi}_s\big)\bigg)\Psi'(\vartheta)d\vartheta\Bigg\| \\
&\quad+\Bigg\|\int_{\varrho_1}^{\varrho_2}\big(\Psi(\varrho_2)-\Psi(\varrho_1)\big)^{\eta-1}\mathcal{Q}_\Psi^\eta(\varrho_2,\vartheta)\mathtt{B}v(\vartheta)\Psi'(\vartheta)d\vartheta\Bigg\| \\
&\quad+\Bigg\|\int_0^{\varrho_1}\Big[\big(\Psi(\varrho_2)-\Psi(\varrho_1)\big)^{\eta-1}-\big(\Psi(\varrho_1)-\Psi(\vartheta)\big)^{\eta-1}\Big]\mathcal{Q}_\Psi^\eta(\varrho_2,\vartheta)\mathtt{B}v(\vartheta)\Psi'(\vartheta)d\vartheta\Bigg\| \\
&\quad+\Bigg\|\int_0^{\varrho_1}\big(\Psi(\varrho_1)-\Psi(\vartheta)\big)^{\eta-1}\Big[\mathcal{Q}_\Psi^\eta(\varrho_2,\vartheta)-\mathcal{Q}_\Psi^\eta(\varrho_1,\vartheta)\Big]\mathtt{B}v(\vartheta)\Psi'(\vartheta)d\vartheta\Bigg\|\Bigg] \\
&=\sum_{i=6}^{11}I_i.
\end{aligned}
$$

From Lemma 4, we obtain

$$
I_6 \leq \big(\Psi(b)-\Psi(0)\big)^{(1-\eta)(1-\xi)}\mathtt{L}''L_{\mathtt{G},\mathtt{r}}(b)\Lambda\big(\mathtt{r}'+E_0(1+\mathtt{r}')\big)\big(\Psi(\varrho_2)-\Psi(\varrho_1)\big)^\eta
$$

and

$$
I_7 \leq \big(\Psi(b)-\Psi(0)\big)^{(1-\eta)(1-\xi)}\mathtt{L}''L_{\mathtt{G},\mathtt{r}}(b)\Lambda\big(\mathtt{r}'+E_0(1+\mathtt{r}')\big)\Big[\big(\Psi(\varrho_2)-\Psi(\vartheta)\big)^\eta-\big(\Psi(\varrho_1)-\Psi(\vartheta)\big)^\eta\Big].
$$

Therefore, $I_6 \to 0$, and $I_7 \to 0$ as $\varrho_2 \to \varrho_1$. Let ϵ be the arbitrary small positive, we can write

$$
\begin{aligned}
I_8 \leq & \sup \big(\Psi(b)-\Psi(0)\big)^{(1-\eta)(1-\xi)}\bigg[\int_0^{\varrho_1-\epsilon}\big(\Psi(\varrho_1)-\Psi(\vartheta)\big)^{\eta-1}\big[\mathcal{Q}_{\Psi}^{\eta}(\varrho_2,\vartheta)-\mathcal{Q}_{\Psi}^{\eta}(\varrho_1,\vartheta)\big] \\
& \times \mathtt{G}\bigg(\vartheta, y_\vartheta+\widehat{\Phi}_\vartheta, \int_0^{\varrho} e\big(\vartheta,s,y_s+\widehat{\Phi}_s\big)\bigg)\Psi'(\vartheta)d\vartheta \\
& + \int_{\varrho_1-\epsilon}^{\varrho_1}\big(\Psi(\varrho_1)-\Psi(\vartheta)\big)^{\eta-1}\big[\mathcal{Q}_{\Psi}^{\eta}(\varrho_2,\vartheta)-\mathcal{Q}_{\Psi}^{\eta}(\varrho_1,\vartheta)\big] \\
& \times \mathtt{G}\bigg(\vartheta, y_\vartheta+\widehat{\Phi}_\vartheta, \int_0^{\varrho} e\big(\vartheta,s,y_s+\widehat{\Phi}_s\big)\bigg)\Psi'(\vartheta)d\vartheta\bigg] \\
\leq & \big(\Psi(b)-\Psi(0)\big)^{(1-\eta)(1-\xi)} L_{\mathtt{G},\mathtt{r}}(b)\Lambda\big(\mathtt{r}'+E_0(1+\mathtt{r}')\big)\int_0^{\varrho_1-\epsilon}\big(\Psi(\varrho_1)-\Psi(\vartheta)\big)^{\eta-1}\Psi'(\vartheta)d\vartheta \\
& \times \sup_{\vartheta\in[0,\varrho_1-\epsilon]}\big\|\mathcal{Q}_{\Psi}^{\eta}(\varrho_2,\vartheta)-\mathcal{Q}_{\Psi}^{\eta}(\varrho_1,\vartheta)\big\| \\
& + \big(\Psi(b)-\Psi(0)\big)^{(1-\eta)(1-\xi)}\mathtt{L}'' L_{\mathtt{G},\mathtt{r}}(b)\Lambda\big(\mathtt{r}'+E_0(1+\mathtt{r}')\big)\int_{\varrho_1-\epsilon}^{\varrho_1}\big(\Psi(\varrho_1)-\Psi(\vartheta)\big)^{\eta-1}\Psi'(\vartheta)d\vartheta.
\end{aligned}
$$

From Lemma 4, we obtain $I_8 \to 0$ as $\varrho_2 \to \varrho_1$ and $\epsilon \to 0$. Using a similar procedure, we obtain that I_9, I_{10} *and* I_{11} tend to zero.

We need to show that, for any $\varrho \in [0,b]$, $\Phi_2'(\varrho) = \big\{(\Phi_2' y)(\varrho) : y \in \mathtt{S}_{\mathtt{r}}\big\}$ is relatively compact in Y.

Take $0 \leq \varrho \leq b$; then, for every $\epsilon > 0$ and $\delta > 0$, let $y \in \mathtt{S}_{\mathtt{r}}$ and define the operator $\Phi_2'^{\epsilon,\delta}$ on $\mathtt{S}_{\mathtt{r}}$ by

$$
\begin{aligned}
\big(\Phi_2'^{\epsilon,\delta} y\big)(\varrho) = & \ \eta\int_0^{\varrho-\epsilon}\int_\delta^\infty \theta\zeta_\eta(\theta)\big(\Psi(\varrho)-\Psi(\vartheta)\big)^{\eta-1} T\big(\big(\Psi(\varrho)-\Psi(0)\big)^{\eta}\theta\big) \\
& \times \mathtt{G}\bigg(\varrho, y_\varrho+\widehat{\Phi}_\varrho, \int_0^{\varrho} e\big(\varrho,s,y_s+\widehat{\Phi}_s\big)\bigg)\Psi'(\vartheta)d\theta d\vartheta \\
& + \eta\int_0^{\varrho-\epsilon}\int_\delta^\infty \theta\zeta_\eta(\theta)\big(\Psi(\varrho)-\Psi(\vartheta)\big)^{\eta-1} T\big(\big(\Psi(\varrho)-\Psi(0)\big)^{\eta}\theta\big)\mathtt{B}v(\vartheta)\Psi'(\vartheta)d\theta d\vartheta \\
= & \ \eta\int_0^{\varrho-\epsilon}\int_\delta^\infty \theta\zeta_\eta(\theta)\big(\Psi(\varrho)-\Psi(\vartheta)\big)^{\eta-1} T\big(\big(\Psi(\varrho)-\Psi(0)\big)^{\eta}\theta+\epsilon^\eta\delta-\epsilon^\eta\delta\big) \\
& \times \bigg[\mathtt{G}\bigg(\varrho, y_\varrho+\widehat{\Phi}_\varrho, \int_0^{\varrho} e\big(\varrho,s,y_s+\widehat{\Phi}_s\big)\bigg)+\mathtt{B}v(\vartheta)\bigg]\Psi'(\vartheta)d\theta d\vartheta \\
= & \ \eta T\big(\epsilon^\eta\delta\big)\int_0^{\varrho-\epsilon}\int_\delta^\infty \theta\zeta_\eta(\theta)\big(\Psi(\varrho)-\Psi(\vartheta)\big)^{\eta-1} T\big(\big(\Psi(\varrho)-\Psi(0)\big)^{\eta}\theta-\epsilon^\eta\delta\big) \\
& \times \bigg[\mathtt{G}\bigg(\varrho, y_\varrho+\widehat{\Phi}_\varrho, \int_0^{\varrho} e\big(\varrho,s,y_s+\widehat{\Phi}_s\big)\bigg)+\mathtt{B}v(\vartheta)\bigg]\Psi'(\vartheta)d\theta d\vartheta.
\end{aligned}
$$

Then, by compactness of $T\big(\epsilon^\eta\delta\big)$ for $\epsilon^\eta\delta > 0$, we obtain that $\Phi_2'^{\epsilon,\delta}(\varrho) = \big\{\big(\Phi^{\epsilon,\delta} y\big)(\varrho) : y \in \mathtt{S}_{\mathtt{r}}\big\}$ is relatively compact in Y. Furthermore, for any $\mathfrak{u} \in \mathtt{S}_p$, we obtain

$$
\begin{aligned}
&\left\|(\Phi_2' y)(\varrho) - (\Phi_2'^{\epsilon,\delta} y)(\varrho)\right\| \\
&\leq \sup\,(\Psi(\rho)-\Psi(0))^{(1-\eta)(1-\xi)}\eta \left\| \int_0^{\varrho}\int_0^{\delta} \theta\zeta_\eta(\theta)(\Psi(\varrho)-\Psi(\vartheta))^{\eta-1} T\big((\Psi(\varrho)-\Psi(0))^{\eta}\theta\big)\right. \\
&\quad \times \left.\left[\mathtt{G}\left(\varrho, y_\varrho + \widehat{\Phi}_\varrho, \int_0^{\varrho} e(\varrho, s, y_s + \widehat{\Phi}_s)\right) + \mathtt{B}v(\vartheta)\right]\Psi'(\vartheta)d\theta d\vartheta \right\| \\
&\quad + \sup\,(\Psi(\rho)-\Psi(0))^{(1-\eta)(1-\xi)}\eta \left\| \int_{\varrho-\epsilon}^{\varrho}\int_{\delta}^{\infty} \theta\zeta_\eta(\theta)(\Psi(\varrho)-\Psi(\vartheta))^{\eta-1} T\big((\Psi(\varrho)-\Psi(0))^{\eta}\theta\big)\right. \\
&\quad \times \left.\left[\mathtt{G}\left(\varrho, y_\varrho + \widehat{\Phi}_\varrho, \int_0^{\varrho} e(\varrho, s, y_s + \widehat{\Phi}_s)\right) + \mathtt{B}v(\vartheta)\right]\Psi'(\vartheta)d\theta d\vartheta \right\| \\
&\leq (\Psi(b)-\Psi(0))^{(1-\eta)(1-\xi)} \Big[M_\eta \big[L_{\mathtt{G},\mathtt{r}}(b)\Lambda(\mathtt{r}' + E_0(1+\mathtt{r}')) \\
&\quad + M_{\mathtt{B}}\|v\|\big] (\Psi(\varrho)-\Psi(0))^{\eta} \left(\int_0^{\delta}\theta\zeta_\eta(\theta)d\theta\right) \\
&\quad + M_\eta\big[L_{\mathtt{G},\mathtt{r}}(b)\Lambda(\mathtt{r}' + E_0(1+\mathtt{r}')) + M_{\mathtt{B}}\|v\|\big](\Psi(\varrho)-\Psi(\varrho-\epsilon))^{\eta}\left(\int_0^{\infty}\theta\zeta_\eta(\theta)d\theta\right)\Big] \\
&\leq (\Psi(b)-\Psi(0))^{(1-\eta)(1-\xi)} \Big[M_\eta \big[L_{\mathtt{G},\mathtt{r}}(b)\Lambda(\mathtt{r}' + E_0(1+\mathtt{r}')) \\
&\quad + M_{\mathtt{B}}\|v\|\big] (\Psi(b)-\Psi(0))^{\eta} \left(\int_0^{\delta}\theta\zeta_\eta(\theta)d\theta\right) \\
&\quad + \left[\frac{M_\eta L_{\mathtt{G},\mathtt{r}}(b)\Lambda(\mathtt{r}' + E_0(1+\mathtt{r}')) + M_{\mathtt{B}}\|v\|}{\Gamma(\eta+1)}\right](\Psi(\varrho)-\Psi(\varrho-\epsilon))^{\eta}\Big],
\end{aligned}
$$

where $\int_0^{\infty}\theta\zeta_\eta(\theta)d\theta = \frac{1}{\Gamma(\eta+1)}$. From the absolute continuity of the Lebesgue integral, we obtain

$$
\left\|(\Phi_2' y)(\varrho) - (\Phi_2'^{\epsilon,\delta} y)(\varrho)\right\| \to 0 \; as \; \epsilon, \delta \to 0.
$$

Thus, there is a relatively compact set that is arbitrarily close to the set $\Phi_2'(\varrho)$ for $\varrho > 0$. Therefore, from the Arzela–Ascoli theorem, it can be observed that $\Phi_2'(\varrho)$ is relatively compact in Y. Hence, the Krasnoselskii fixed point theorem (Lemma 6) Φ has a fixed point in $\mathtt{S}_{\mathtt{r}}$, which is the mild solution of the system (1). □

Here, we focus on the approximate controllability of Equation (1).

Theorem 2. *Suppose that* (H_1)–(H_5) *hold and* $\mathtt{G}$ *and* $\mathtt{H}$ *are a uniformly bounded function. Furthermore, the corresponding linear Equation (8) is approximately controllable on* $\mathcal{I}$*; then, system (1) is approximately controllable on* $\mathcal{I}$.

Proof. Let u^{λ} be a fixed point of Φ in $\mathtt{S}_{\mathtt{r}}$; using Theorem 1, any fixed point u^{λ} is a mild solution of system (1), such that

$$
\begin{aligned}
\mathtt{u}^{\lambda}(\varrho) &= \mathcal{S}_{\Psi}^{\eta,\xi}(\varrho,0)\big[\phi_0 - \mathtt{H}(0,\mathtt{u}(0))\big] + \mathtt{H}\big(\rho,\mathtt{u}_{\rho}^{\lambda}\big) + \int_0^{\varrho}\big(\Psi(\varrho)-\Psi(\vartheta)\big)^{\eta-1}\mathtt{A}\mathcal{Q}_{\Psi}^{\eta}(\varrho,\vartheta)\mathtt{H}\big(\vartheta,\mathtt{u}_{\vartheta}^{\lambda}\big)\Psi'(\vartheta)d\vartheta \\
&+ \int_0^{\varrho}\big(\Psi(\varrho)-\Psi(\vartheta)\big)^{\eta-1}\mathcal{Q}_{\Psi}^{\eta}(\varrho,\vartheta)\mathtt{BB}^*\mathcal{Q}_{\Psi}^{\eta *}(b,\rho)\mathtt{R}\big(\gamma,\mathfrak{T}_0^b\big) \\
&\times\Big[\mathtt{u}_1 - \big(\Psi(b)-\Psi(0)\big)^{(1-\eta)(\xi-1)}\Big[\mathcal{S}_{\Psi}^{\eta,\xi}(\varrho,0)\big[\phi_0 - \mathtt{H}(0,\mathtt{u}(0))\big] + \mathtt{H}\big(\rho,\mathtt{u}_{\rho}^{\lambda}\big) \\
&+ \int_0^{b}\big(\Psi(b)-\Psi(\delta)\big)^{\eta-1}\mathtt{A}\mathcal{Q}_{\Psi}^{\eta}(b,\delta)\mathtt{H}\big(\delta,\mathtt{u}_{\delta}^{\lambda}\big)\Psi'(\delta)d\delta \\
&+ \int_0^{b}\big(\Psi(b)-\Psi(\delta)\big)^{\eta-1}\mathcal{Q}_{\Psi}^{\eta}(b,\delta)\mathtt{G}\Big(\omega,\mathtt{u}_{\omega}^{\lambda},\int_0^{\omega}e\big(\omega,s,\mathtt{u}_s^{\lambda}\big)ds\Big)\Psi'(\delta)d\delta\Big]\Big]\Psi'(\vartheta)d\vartheta \\
&+ \int_0^{\varrho}\big(\Psi(\varrho)-\Psi(\vartheta)\big)^{\eta-1}\mathcal{Q}_{\Psi}^{\eta}(\varrho,\vartheta)\mathtt{G}\Big(\varrho,\mathtt{u}_{\varrho}^{\lambda},\int_0^{\varrho}e\big(\varrho,s,\mathtt{u}_s^{\lambda}\big)ds\Big)\Psi'(\vartheta)d\vartheta
\end{aligned}
$$

Define

$$
\begin{aligned}
P(\mathtt{u}^{\lambda}) = \mathtt{u}_1 - \Big[&\mathcal{S}_{\Psi}^{\eta,\xi}(\varrho,0)\big[\phi_0 - \mathtt{H}(0,\mathtt{u}(0))\big] + \mathtt{H}\big(\rho,\mathtt{u}_{\rho}^{\lambda}\big) \\
&+ \int_0^{b}\big(\Psi(b)-\Psi(\delta)\big)^{\eta-1}\mathtt{A}\mathcal{Q}_{\Psi}^{\eta}(b,\delta)\mathtt{H}\big(\delta,\mathtt{u}_{\delta}^{\lambda}\big)\Psi'(\delta)d\delta \\
&+ \int_0^{b}\big(\Psi(b)-\Psi(\delta)\big)^{\eta-1}\mathcal{Q}_{\Psi}^{\eta}(b,\delta)\mathtt{G}\Big(\omega,\mathtt{u}_{\omega}^{\lambda},\int_0^{\omega}e\big(\omega,s,\mathtt{u}_s^{\lambda}\big)ds\Big)\Psi'(\delta)d\delta\Big]
\end{aligned}
$$

We have $\big(I - \mathfrak{T}_0^b\mathtt{R}\big(\gamma,\mathfrak{T}_0^b\big)\big) = \lambda\mathtt{R}(\lambda,\mathfrak{T}_0^b)$, then

$$
\begin{aligned}
\mathtt{u}^{\lambda}(b) &= \mathcal{S}_{\Psi}^{\eta,\xi}(b,0)\big[\phi_0 - \mathtt{H}(0,\mathtt{u}(0))\big] + \mathtt{H}\big(b,\mathtt{u}_{b}^{\lambda}\big) + \int_0^{b}\big(\Psi(b)-\Psi(\vartheta)\big)^{\eta-1}\mathtt{A}\mathcal{Q}_{\Psi}^{\eta}(b,\vartheta)\mathtt{H}\big(\vartheta,\mathtt{u}_{\vartheta}^{\lambda}\big)\Psi'(\vartheta)d\vartheta \\
&+ \int_0^{b}\big(\Psi(b)-\Psi(\vartheta)\big)^{\eta-1}\mathcal{Q}_{\Psi}^{\eta}(b,\vartheta)\mathtt{BB}^*\mathcal{Q}_{\Psi}^{\eta *}(b,\rho)\mathtt{R}\big(\lambda,\mathfrak{T}_0^b\big) \\
&\times\Big[\mathtt{u}_1 - \mathcal{S}_{\Psi}^{\eta,\xi}(\varrho,0)\big[\phi_0 - \mathtt{H}(0,\mathtt{u}(0))\big] - \mathtt{H}\big(b,\mathtt{u}_{b}^{\lambda}\big) \\
&- \int_0^{b}\big(\Psi(b)-\Psi(\delta)\big)^{\eta-1}\mathtt{A}\mathcal{Q}_{\Psi}^{\eta}(b,\delta)\mathtt{H}\big(\delta,\mathtt{u}_{\delta}^{\lambda}\big)\Psi'(\delta)d\delta \\
&- \int_0^{b}\big(\Psi(b)-\Psi(\delta)\big)^{\eta-1}\mathcal{Q}_{\Psi}^{\eta}(b,\delta)\mathtt{G}\Big(\omega,\mathtt{u}_{\omega}^{\lambda},\int_0^{\omega}e\big(\omega,s,\mathtt{u}_s^{\lambda}\big)ds\Big)\Psi'(\delta)d\delta\Big]\Psi'(\vartheta)d\vartheta \\
&+ \int_0^{b}\big(\Psi(\varrho)-\Psi(\vartheta)\big)^{\eta-1}\mathcal{Q}_{\Psi}^{\eta}(\varrho,\vartheta)\mathtt{G}\Big(\varrho,\mathtt{u}_{\varrho}^{\lambda},\int_0^{\varrho}e\big(\varrho,s,\mathtt{u}_s^{\lambda}\big)ds\Big)\Psi'(\vartheta)d\vartheta \\
&= \mathcal{S}_{\Psi}^{\eta,\xi}(b,0)\big[\phi_0 - \mathtt{H}(0,\mathtt{u}(0))\big] + \mathtt{H}\big(b,\mathtt{u}_{b}^{\lambda}\big) + \int_0^{b}\big(\Psi(b)-\Psi(\vartheta)\big)^{\eta-1}\mathtt{A}\mathcal{Q}_{\Psi}^{\eta}(b,\vartheta)\mathtt{H}\big(\vartheta,\mathtt{u}_{\vartheta}\big)\Psi'(\vartheta)d\vartheta \\
&+ \int_0^{b}\big(\Psi(\varrho)-\Psi(\vartheta)\big)^{\eta-1}\mathcal{Q}_{\Psi}^{\eta}(\varrho,\vartheta)\mathtt{G}\Big(\varrho,\mathtt{u}_{\varrho}^{\lambda},\int_0^{\varrho}e\big(\varrho,s,\mathtt{u}_s^{\lambda}\big)ds\Big)\Psi'(\vartheta)d\vartheta \\
&+ \mathfrak{T}_0^b\mathtt{R}\big(\lambda,\mathfrak{T}_0^b\big)P(\mathtt{u}^{\lambda}) \\
&= \mathcal{S}_{\Psi}^{\eta,\xi}(b,0)\big[\phi_0 - \mathtt{H}(0,\mathtt{u}(0))\big] + \mathtt{H}\big(b,\mathtt{u}_{b}^{\lambda}\big) + \int_0^{b}\big(\Psi(b)-\Psi(\vartheta)\big)^{\eta-1}\mathtt{A}\mathcal{Q}_{\Psi}^{\eta}(b,\vartheta)\mathtt{H}\big(\vartheta,\mathtt{u}_{\vartheta}\big)\Psi'(\vartheta)d\vartheta \\
&+ \int_0^{b}\big(\Psi(\varrho)-\Psi(\vartheta)\big)^{\eta-1}\mathcal{Q}_{\Psi}^{\eta}(\varrho,\vartheta)\mathtt{G}\Big(\varrho,\mathtt{u}_{\varrho}^{\lambda},\int_0^{\varrho}e\big(\varrho,s,\mathtt{u}_s^{\lambda}\big)ds\Big)\Psi'(\vartheta)d\vartheta \\
&+ P(\mathtt{u}^{\lambda}) - \lambda\mathtt{R}\big(\lambda,\mathfrak{T}_0^b\big)P(\mathtt{u}^{\lambda}) \\
&= \mathtt{u}_1 - \alpha\mathtt{R}\big(\lambda,\mathfrak{T}_0^b\big)P(\mathtt{u}^{\lambda}).
\end{aligned}
$$

Using Dunford–Pettis theorem, there is a subsequence $\left\{ \mathtt{G}\left(\delta, \mathtt{u}_\delta^\lambda, \int_0^\varrho e\left(\delta, s, \mathtt{u}_s^\lambda\right)\right) \right\}$ that converges weakly to $\left\{ \mathtt{G}\left(\delta, \mathtt{u}_\delta, \int_0^\varrho e\left(\delta, s, \mathtt{u}_s\right) ds\right) \right\}$ in $L^1(\mathcal{I}, Y)$, and similarly, $\{\mathtt{H}(\delta, \mathtt{u}_\delta^\lambda)\}$ also converges.

Consider the following:

$$\begin{aligned} \mathtt{W} = \mathtt{u}_1 - &\Bigg[\mathcal{S}_\Psi^{\eta,\xi}(\varrho, 0)\left[\phi_0 - \mathtt{H}(0, \mathtt{u}(0))\right] - \mathtt{H}\left(\rho, \mathtt{u}_\rho\right) \\ &- \int_0^b \left(\Psi(b) - \Psi(\delta)\right)^{\eta-1} \mathtt{A}\mathcal{Q}_\Psi^\eta(b, \delta) \mathtt{H}\left(\delta, \mathtt{u}_\delta\right) \Psi'(\delta) d\delta \\ &- \int_0^b \left(\Psi(b) - \Psi(\delta)\right)^{\eta-1} \mathcal{Q}_\Psi^\eta(b, \delta) \mathtt{G}\left(\delta, \mathtt{u}_\delta, \int_0^\varrho e\left(\delta, s, \mathtt{u}_s\right) ds\right) \Psi'(\delta) d\delta \Bigg]. \end{aligned}$$

We obtain

$$\begin{aligned} &\left\| P(\mathtt{u}^\gamma) - \mathtt{W} \right\| \\ &= \Bigg\| \sup \left(\Psi(b) - \Psi(0)\right)^{(1-\eta)(1-\xi)} \Bigg[\mathtt{H}\left(b, \mathtt{u}_b^\lambda\right) + \int_0^b \left(\Psi(b) - \Psi(\delta)\right)^{\eta-1} \mathtt{A}\mathcal{Q}_\Psi^\eta(b, \delta) \mathtt{H}\left(\delta, \mathtt{u}_\delta^\lambda\right) \Psi'(\delta) d\delta \\ &\quad + \int_0^b \left(\Psi(b) - \Psi(\delta)\right)^{\eta-1} \mathcal{Q}_\Psi^\eta(b, \delta) \mathtt{G}\left(\delta, \mathtt{u}_\delta^\lambda, \int_0^\varrho e\left(\delta, s, \mathtt{u}_s^\lambda\right) ds\right) \Psi'(\delta) d\delta \\ &\quad - \Bigg(\mathtt{H}\left(\rho, \mathtt{u}_\rho\right) + \int_0^b \left(\Psi(b) - \Psi(\delta)\right)^{\eta-1} \mathtt{A}\mathcal{Q}_\Psi^\eta(b, \delta) \mathtt{H}\left(\delta, \mathtt{u}_\delta\right) \Psi'(\delta) d\delta \\ &\quad + \int_0^b \left(\Psi(b) - \Psi(\delta)\right)^{\eta-1} \mathcal{Q}_\Psi^\eta(b, \delta) \mathtt{G}\left(\delta, \mathtt{u}_\delta, \int_0^\varrho e\left(\delta, s, \mathtt{u}_s\right) ds\right) \Psi'(\delta) d\delta \Bigg) \Bigg] \Bigg\| \\ &\leq \left(\Psi(b) - \Psi(0)\right)^{(1-\eta)(1-\xi)} \Bigg[\left\| \mathtt{H}\left(b, \mathtt{u}_b^\lambda\right) - \mathtt{H}\left(\rho, \mathtt{u}_\rho\right) \right\| \\ &\quad + \int_0^b \left(\Psi(b) - \Psi(\delta)\right)^{\eta-1} \mathtt{A}\mathcal{Q}_\Psi^\eta(b, \delta) \left\| \mathtt{H}\left(\delta, \mathtt{u}_\delta^\lambda\right) - \mathtt{H}\left(\delta, \mathtt{u}_\delta\right) \right\| \Psi'(\delta) d\delta \\ &\quad + \int_0^b \left(\Psi(b) - \Psi(\delta)\right)^{\eta-1} \mathcal{Q}_\Psi^\eta(b, \delta) \left\| \mathtt{G}\left(\delta, \mathtt{u}_\delta^\lambda, \int_0^\varrho e\left(\delta, s, \mathtt{u}_s^\lambda\right) ds\right) - \mathtt{G}\left(\delta, \mathtt{u}_\delta, \int_0^\varrho e\left(\delta, s, \mathtt{u}_s\right) ds\right) \right\| \Psi'(\delta) d\delta \Bigg] \end{aligned}$$

From the uniform boundedness of $\{\mathtt{G}^\lambda(\cdot, \cdot, \cdot)\}$ and $\{\mathtt{H}^\lambda(\cdot, \cdot)\}$, there exists $\mathtt{G}$, $\mathtt{H} \in L^1(\mathcal{I}, Y)$, such that

$$\begin{gathered} \mathtt{G}\left(\delta, \mathtt{u}_\delta^\lambda, \int_0^\omega e\left(\delta, s, \mathtt{u}_s^\lambda\right) ds\right) \to \mathtt{G}\left(\delta, \mathtt{u}_\delta, \int_0^\omega e\left(\delta, s, \mathtt{u}_s\right) ds\right) \ as\ \lambda \to 0, \\ \mathtt{H}\left(\delta, \mathtt{u}_\delta^\lambda\right) \to \mathtt{H}\left(\delta, \mathtt{u}_\delta\right) \ as\ \lambda \to 0. \end{gathered}$$

Furthermore, approximating controllability of system (8), we obtain $\lambda \mathtt{R}(\lambda, \mathfrak{T}_0^b) \to 0$ as $\lambda \to 0^+$ in the strong continuous topology. Thus, we can obtain that as $\lambda \to 0^+$,

$$\begin{aligned} \left\| \mathtt{u}^\lambda(b) - \mathtt{u}_1 \right\| &\leq \left\| \lambda \mathtt{R}\left(\lambda, \mathfrak{T}_0^b\right)(\mathtt{W}) \right\| + \left\| \lambda \mathtt{R}\left(\lambda, \mathfrak{T}_0^b\right)\left(P(\mathtt{u}^\lambda) - \mathtt{W}\right) \right\| \\ &\leq \left\| \lambda \mathtt{R}\left(\lambda, \mathfrak{T}_0^b\right) \mathtt{W} \right\| + \left\| \left(P(\mathtt{u}^\lambda) - \mathtt{W}\right) \right\| \to 0. \end{aligned}$$

Hence, system (1) is approximately controllable on $\mathcal{I}$. □

4. Application

4.1. Application 1

Observe these systems of Ψ-HFD_{tial} with infinite delay:

$$
\begin{aligned}
{}^{H}D^{\frac{2}{3},\xi;\Psi}\left[\mathtt{u}(\varrho,\sigma)+\int_0^{\pi}\mathtt{H}(z,\sigma)\mathtt{u}(\rho,z)dz\right] &= \frac{\partial^2}{\partial\sigma^2}\mathtt{u}(\varrho,\sigma)+\mathtt{W}\mu(\varrho,\sigma)\\
&+\mathsf{G}\left(\varrho,\int_{-\infty}^{\varrho}\mathsf{G}_1(\omega-\varrho)\mathtt{u}(\omega,\sigma)d\omega,\int_0^{\varrho}\int_{-\infty}^{0}\mathsf{G}_2(\omega,\sigma,r-\omega)\mathtt{u}(r,\sigma)d\omega d\sigma\right),
\end{aligned} \tag{15}
$$

$$
\begin{aligned}
&\mathtt{u}(0,\sigma)=\mathtt{u}_0(\tau),\ \sigma\in[0,\pi],\\
&\mathtt{u}(\varrho,0)=\mathtt{u}(\varrho,\pi)=0,\ \varrho\in\mathcal{I},\\
&\mathtt{u}(\varrho,\sigma)=\phi(\varrho,\sigma),\ 0\le\sigma\le\pi,\ \varrho\in(-\infty,0],
\end{aligned}
$$

Here, ${}^{H}D^{\frac{2}{3},\xi;\Psi}$ is the Ψ-HFD_{ve} of order $\frac{2}{3}$, $\mathsf{G}:\mathcal{I}\times\mathtt{S}_w\times Y\to Y$ is a continuous function, and G_1 and G_2 are the required functions. Let $Y=L^2([0,\pi])$ be endowed with the usual norm $\|\cdot\|_{L^2}$, and define the operator $\mathtt{A}:D(\mathtt{A})\subset Y\to Y$ by

$$
D(\mathtt{A})=\{\mathfrak{a}\in Y:\mathfrak{a},\mathfrak{a}'\text{ are absolutely continuous and }\mathfrak{a}''\in Y,\mathfrak{a}(0)=\mathfrak{a}(\pi)=0\},
$$

and $\mathtt{A}\mathtt{u}=\frac{\partial^2}{\partial y^2}$. Also, we can observe that $\mathtt{A}$ has a discrete spectrum; the eigenvalues are $m^2, m\in\mathbb{N}$, with the eigen vectors $\mathtt{e}_m(z)=\sqrt{\frac{2}{\pi}}\sin(mz)$.

Furthermore, the infinitesimal generator $\mathtt{A}$ generates a uniformly bounded analytic semigroup $\{\mathtt{T}(\varrho)\}_{\varrho\ge0}$ on Y, i.e.,

$$
\mathtt{T}(\varrho)\mathfrak{a}=\sum_{m=1}^{\infty}e^{-m^2\varrho}\langle\mathfrak{a},e_m\rangle e_m,\quad \mathfrak{a}\in Y,
$$

where $\|\mathtt{T}(\varrho)\|\le e^{-\varrho}$ for all $\varrho\ge0$. Therefore, we give $\kappa_\eta=1$, which implies that $\sup_{\varrho\in[0,\infty)}\|\mathtt{T}(\varrho)\|=1$, and hypotheses (H_1) is satisfied. Take

$$
\begin{aligned}
\mathtt{u}(\varrho)(\sigma)&=\mathtt{u}(\varrho,\sigma),\\
v(\varrho)(\sigma)&=\phi(\varrho,\sigma),\\
\mathtt{G}\left(\varrho,\mathtt{u}_\varrho,\int_0^{\varrho}e(\varrho,s,\mathtt{u}_s)ds\right)&=\mathsf{G}\left(\varrho,\int_{-\infty}^{\varrho}\mathsf{G}_1(\omega-\varrho)\mathtt{u}(\omega,\sigma)d\omega,\int_0^{\varrho}\int_{-\infty}^{0}\mathsf{G}_2(\omega,\sigma,r-\omega)\mathtt{u}(r,\sigma)d\omega d\sigma\right),\\
\int_0^{\varrho}e(\varrho,s,\mathtt{u}_s)ds&=\int_0^{\varrho}\int_{-\infty}^{0}\mathsf{G}_2(\omega,\sigma,r-\omega)\mathtt{u}(r,\sigma)d\omega d\sigma.
\end{aligned}
$$

Suppose $w(\theta)=exp(2\theta),\ \theta<0$; then, $\int_{-\infty}^{0}w(\theta)d\theta=\frac{1}{2}$, and we must obtain:

$$
\|\delta\|_{\mathrm{Y}}=\int_{-\infty}^{0}(\Psi(\theta)-\Psi(0))^{(1-\eta)(1-\xi)}w(\theta)\|\delta(\theta)\|_{[-n,0]}d\theta,
$$

We can make a Banach space $(\mathtt{S}'_w,\|\cdot\|_{\mathrm{Y}})$ and satisfy Lemma 1. Also, the corresponding functions F, F_1, and F_2 are satisfied $(H_2),(H_3)$.

Take $\Psi(\varrho)=\sqrt{\varrho+1},\ \kappa_\eta=1$; then, we obtain (9):

$$
\frac{\mathsf{M}}{\Gamma(\frac{5}{3})}(\sqrt{2}-1)^{2/3}<1.
$$

Hence, according to Theorem 1, system (1) has a mild solution on $[0,1]$.

Here, let $\mathtt{B}:U\to U$ be an operator with $U=L^2([0,\pi])$, defined by

$$
(\mathtt{B}v)(\varrho)(y)=W\mu(y,\varrho),\ 0<y<\pi.
$$

With the choice of $\mathtt{A}$, $\mathtt{B}$, and $\mathtt{G}$, system (15) can be expressed as

$$ {}^{H}D^{\frac{2}{3},\xi;\Psi}\left[\mathtt{u}(\varrho)-\mathtt{H}(\rho,\mathtt{u}_{\rho})\right]=\mathtt{A}\mathtt{u}(\varrho)+\mathtt{G}\left(\varrho,\mathtt{u}_{\varrho},\int_{0}^{\varrho}e(\varrho,s,\mathtt{u}_{s})ds\right)+\mathtt{B}v(\varrho),\ \varrho\in(0,1], $$
$$ I_{0^{+}}^{(1-\eta)(1-\xi)}\mathtt{u}(0)=\phi_{0}, \tag{16} $$

Thus, the assumptions (H_1)–(H_5) are satisfied. Furthermore, the linear system (8) corresponding to (15) is approximately controllable and satisfies Theorem 1. Therefore, the corresponding system (15) obeys Theorem 2; hence, it is approximately controllable.

4.2. Application 2

In this part, we examine the Hilfer-fractional-differential-equation-based IVP and demonstrate how fractional derivatives with respect to another function might be advantageous. Consider the mild solution of system (1),

$$ \begin{aligned} \mathtt{u}(\varrho)=&\,\mathcal{S}_{\Psi}^{\eta,\xi}(\varrho,0)\left[\phi_{0}-\mathtt{H}(0,\mathtt{u}(0))\right]+\mathtt{H}(\rho,\mathtt{u}_{\rho})+\int_{0}^{\varrho}\left(\Psi(\varrho)-\Psi(\vartheta)\right)^{\eta-1}\mathtt{A}\mathcal{Q}_{\Psi}^{\eta}(\varrho,\vartheta)\mathtt{H}(\vartheta,\mathtt{u}_{\vartheta})\Psi'(\vartheta)d\vartheta\\ &+\int_{0}^{\varrho}\left(\Psi(\varrho)-\Psi(\vartheta)\right)^{\eta-1}\mathcal{Q}_{\Psi}^{\eta}(\varrho,\vartheta)\mathtt{B}v(\vartheta)\Psi'(\vartheta)d\vartheta\\ &+\int_{0}^{\varrho}\left(\Psi(\varrho)-\Psi(\vartheta)\right)^{\eta-1}\mathcal{Q}_{\Psi}^{\eta}(\varrho,\vartheta)\mathtt{G}\left(\varrho,\mathtt{u}_{\varrho},\int_{0}^{\vartheta}e(\varrho,s,\mathtt{u}_{s})ds\right)\Psi'(\vartheta)d\vartheta,\quad for\ \varrho\in[0,b]. \end{aligned} \tag{17} $$

In the realm of digital signal processing (DSP), digital filters play a very important role. In reality, the way that the digital filter is implemented is exceptional; this is one of the key reasons why DSP is becoming more and more well liked. Typically, we categorise filters based on their two primary uses: signal separation and signal restoration. If a tiny signal is affected together with agitation, sound, disturbance, or other signals, the use of filters in signal separation is crucial, for instance, if there was a gadget that could calculate the electrical activity of a baby's heart (EKG) while it was still in the womb. The gross indication may possibly be influenced by means of the inhalation and pulse of the mother. One can use a filter for segregating these signals with the target that those may be explored individually.

When a signal is distorted in some way, we could use signal restoration. For instance, sound recordings made using the equipment may be separated, especially when it comes to describing the sound's occurrence. Similarly, there is still another technique for advanced channels that is referred to as recursion. Currently, we use convolution to apply a filter; each application in earnings is defined by balancing the models and combining them. Motivated by the filter system presented in [33–35], we present the digital filter system corresponding to the mild solution in (1). Digital filters are the back bone for any signal processing application. Many bio-medical signals related to the human body are, nowadays, acquired for various informative feature extractions. Most of the mentioned signals, in general, possess a low frequency by nature. These signals describe information pertaining to various disorders and diseases for which the accuracy is of high concern. The efficiency of any digital signal processing filtering system relies on the ability to reject noise.

Figure 1 describes the following:

1. Product modulator 1 accepts the input $\mathtt{u}(\rho)$ and $\mathtt{H}(\cdot)$ produces the output $\mathtt{H}(\rho,\mathtt{u}_{\rho})$.
2. Product modulator 2 accepts the input $\mathtt{H}(\cdot,\mathtt{u}_{\rho})$ and $\mathtt{A}$ and gives out put $\mathtt{AH}(\cdot)$.
3. Product modulator 3 accepts the input $\mathcal{Q}_{\Psi}^{\eta}(\rho,\vartheta)$ and $\mathtt{AH}(\cdot,\cdot)$ produces the output $\mathcal{Q}_{\Psi}^{\eta}\mathtt{AH}(\cdot)$.
4. Product modulator 4 accepts the input $\mathcal{Q}_{\Psi}^{\eta}\mathtt{AH}(\cdot,\mathtt{u}_{\rho})$ and Ψ-function, and obtains the output $\left(\Psi(\rho)-\Psi(\vartheta)\right)^{\eta-1}\mathcal{Q}_{\Psi}^{\eta}\mathtt{AH}(\cdot,\mathtt{u}_{\rho})\Psi'(\vartheta)$.
5. Product modulator 5 accepts the input $v(\rho)$ and $\mathtt{B}$, and produces the output $\mathtt{B}v(\rho)$.
6. Product modulator 6 accepts the input $\mathcal{Q}_{\Psi}^{\eta}(\rho,\vartheta)$ and $\mathtt{B}v(\rho)$, and gives the output $\mathtt{B}\mathcal{Q}_{\Psi}^{\eta}(\rho,\vartheta)v(\rho)$.

7. Product modulator 7 accepts the input $\mathtt{u}(\rho)$ and $\mathtt{e}(\cdot)$, and gives the output $e(\cdot, \mathtt{u}_\rho)$ over the period $(0, \rho)$.
8. The integrator executes the input $\mathtt{G}(\cdots)$ and $\int e(\cdot, \mathtt{u}_\rho)$ and produces the output $\mathtt{G}\left(\varrho, \mathtt{u}_\varrho, \int_0^\varrho e(\varrho, s, \mathtt{u}_s)ds\right)$ over the period of time $(0, \rho)$, $\forall\, \rho \in [0, b]$.
9. Product modulator 8 accepts the input $\mathcal{Q}_\Psi^\eta(\rho, \vartheta)$ and $\mathtt{G}\left(\varrho, \mathtt{u}_\varrho, \int_0^\varrho e(\varrho, s, \mathtt{u}_s)ds\right)$ and gives the output $\mathcal{Q}_\Psi^\eta(\rho, \vartheta)\mathtt{G}\left(\varrho, \mathtt{u}_\varrho, \int_0^\varrho e(\varrho, s, \mathtt{u}_s)ds\right)$.
10. Product modulator 9 accepts $[\phi_0 + \mathtt{H}(0, \mathtt{u}(0))]$ and $\mathcal{S}_\Psi^{\eta,\xi}(\rho, 0)$ at time $\rho = 0$, and produces $\mathcal{S}_\Psi^{\eta,\xi}(\rho, 0)[\phi_0 + \mathtt{H}(0, \mathtt{u}(0))]$.
11. The integrators execute the following value: $\left(\Psi(\rho) - \Psi(\vartheta)\right)^{\eta-1}\mathcal{Q}_\Psi^\eta(\rho, \vartheta)\left[\mathtt{A}(\vartheta, \mathtt{u}_\vartheta) + \mathtt{B}v(\rho) + \mathtt{G}\left(\rho, \mathtt{u}_\rho, \int_0^\rho e(\rho, s, \mathtt{u}_s)\right)ds\right]\Psi'(\vartheta)$, and produces the integral value over the period ρ.
    Finally, we turn all outputs from the integrators to the summer network and the output of $\mathtt{u}(\rho)$ is obtained; it is bounded and approximately controllable.

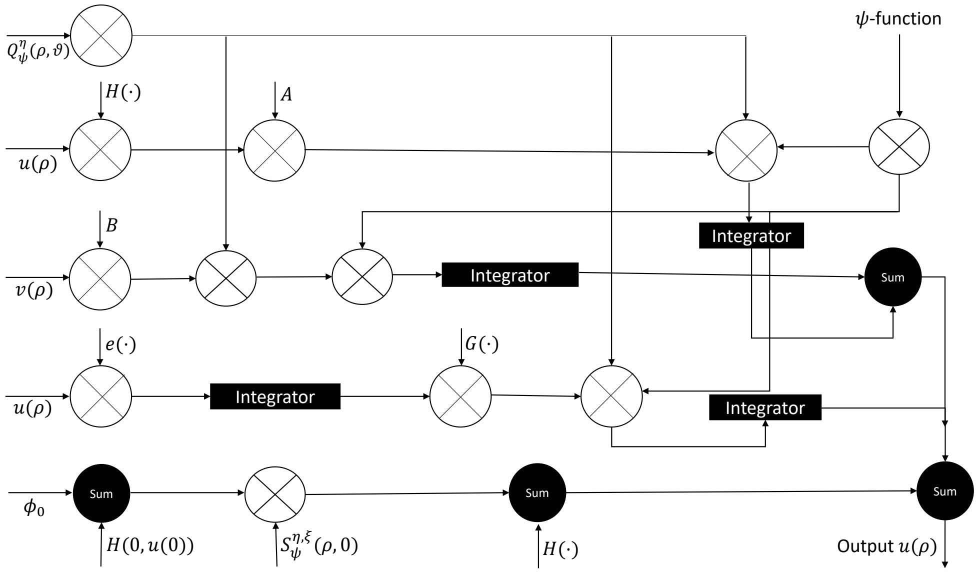

Figure 1. Filter system model.

5. Conclusions

In this work, we studied about the approximate controllability of Ψ-HFD_{tial} equations with infinite delay by using a fixed point method. The major results were established by applying the semigroup theory, Ψ-HFD_{ve}, and fixed point theorem. Two applications (theoretical and filter system) were provided to illustrate the principle. In the future, we will focus on the exact controllability of Ψ-HFD_{tial} systems and real-life applications using fractional differential systems via a fixed point approach.

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Abbreviations

The following abbreviations are used in this manuscript:

HFD_{ve}	Hilfer Fractional Derivative
HFD_{tial}	Hilfer Fractional Differential
MNC	Measure of Noncompactness.

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