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Gravitational and Electromagnetic Perturbations of a Charged Black Hole in a General Gauge Condition

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Abstract: We derive a set of coupled equations for the gravitational and electromagnetic perturbation in the Reissner–Nordström geometry using the Newman–Penrose formalism. We show that the information of the physical gravitational signal is contained in the Weyl scalar function Ψ_4 , as is well known, but for the electromagnetic signal, the information is encoded in the function χ , which relates the perturbations of the radiative Maxwell scalars ϕ_2 and the Weyl scalar Ψ_3 . In deriving the perturbation equations, we do not impose any gauge condition and as a limiting case, our analysis contains previously obtained results, for instance, those from Chandrasekhar’s book. In our analysis, we also include the sources for the perturbations and focus on a dust-like charged fluid distribution falling radially into the black hole. Finally, by writing the functions on the basis of spin-weighted spherical harmonics and the Reissner–Nordström spacetime in Kerr–Schild type coordinates, a hyperbolic system of coupled partial differential equations is presented and numerically solved. In this way, we completely solve a system that generates a gravitational signal as well as an electromagnetic/gravitational one, which sets the basis to find correlations between them and thus facilitates gravitational wave detection via electromagnetic signals.

Keywords: perturbation theory; black holes; gravitational-waves



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1. Introduction

Gravitational wave astronomy was born in 2015 with the discovery of the first astronomical source named GW150914 [1–3]. This was the first observation of a binary black hole system and was done with the gravitational wave interferometer LIGO [4], which detected the gravitational wave signal produced by the merger of two black holes in a binary system. The two black holes were not surrounded by a significant amount of matter that could generate electromagnetic emissions; therefore, in this respect, the emission of the merger was purely in the gravitational channel. Unlike black hole binaries, a system involving neutron stars does possess an electromagnetic counterpart, as demonstrated by the detection of the neutron star merger event in gravitational and electromagnetic messenger channels in 2017, named GW170817 [5,6].

During the formation of an accretion disk around a black hole, electrons may escape from the influence of the central object leaving a net charge in the system, these charged particles may be captured by the black hole, producing a charged black hole [7]. Thus, the study of the electromagnetic and gravitational perturbations can be used to describe the scattering of both types of waves. However, for a charged black hole, a gravitational perturbation of the metric inevitably accompanies a perturbation on the electromagnetic field and vice versa. The description of coupled electromagnetic and gravitational perturbations on charged black holes have been discussed in several studies [8–14] using

different techniques including dispersion of waves due to curvature potentials [15–17] and the Newman–Penrose formalism [18–20].

Chandrasekhar described the scattering of electromagnetic waves on a Reissner–Nordström black hole and the resulting generation of outgoing gravitational waves, and using the gauge freedom of the Maxwell equations in a curved background, he derived the electromagnetic equations by finding a gauge that restores the symmetry to the perturbation equations [21]. He showed that the curved spacetime produced by a black hole is sensitive to the electromagnetic field part χ of the spacetime and this awareness is manifested in the symmetry of the equations for the scalars $\varphi_2^{(1)}$ and $\Psi_3^{(1)}$ in a curved background. With the introduction of this particular gauge, the electromagnetic and gravitational perturbation equations simplify greatly. Furthermore, using this gauge, the equations for the gravitational and electromagnetic Weyl scalars decouple from the rest of the functions appearing in the system of equations. As a result of the apparent cognizance of the curved geometry to the existence of the Maxwell field, in which the symmetry in the equations is recovered, Chandrasekhar dubbed this gauge the phantom gauge. Since then, the scattering of both gravitational and electromagnetic waves have been described using this gauge in a variety of studies [22–24]. In [22], Lee found a pair of equations for only two gauge invariant quantities involving electromagnetic and gravitational perturbations in a Kerr–Newman spacetime without using the phantom gauge. In this work, we revisit the findings of Lee, explicitly including the matter sources that may cause the perturbation in a Reissner–Nordström background.

As a direct application of our setting, we consider a pressureless charged perfect fluid (dust) falling radially into the black hole as the cause of the perturbation. We explicitly write the equations in a coordinate system and expand the functions using a spherical harmonic basis with the appropriate spin weight. With this choice, we show that the dynamics of the perturbed functions $\Psi_4^{(1)}$ and χ are completely determined by a set of partial differential equations that depend on the radial and temporal coordinates only. We numerically solve this set of equations and obtain the waveforms for several representative values of the parameters of the system. This sets the basis for a thorough comparative analysis that might find correlations between the waveforms and thus, by the detection of one of these electromagnetic/gravitational signals, one will be able to infer the presence of a purely gravitational one. This will be carried out in future research.

The paper is structured as follows: In Section 2, we introduce the basics of the Newman–Penrose formalism including the Bianchi and Maxwell identities. In Section 3, we provide a detailed derivation of the perturbation equations in a Reissner–Nordström background. In Section 4, we describe the sources of the perturbation and show that by choosing an adequate decomposition in spin-weighted spherical harmonics, it is possible to separate the time-spatial structure of the equations. In Section 5, we introduce the tetrad and geometric quantities in the Reissner–Nordström background described in horizon penetrating coordinates, present a numerical scheme to solve the equations for a particular scenario of matter falling into the black hole, and present some waveforms: gravitational as well as those related with χ . Finally some concluding remarks are given in Section 6. In the rest of the paper, we use η to indicate the signature of the metric: $\eta = 1$ for signature $(+, -, -, -)$, and $\eta = -1$ for the signature $(-, +, +, +)$, and we use geometric units where $c = G = 1$.

2. Foundations: Newman–Penrose Formalism

The starting point in the Newman–Penrose formalism is to define a tetrad of null vectors [25]. The choice of the tetrad is made to reflect symmetries of spacetime, since certain components may vanish, leading to simplification of field equations. In this work, we use l^μ and k^μ to denote ingoing and outgoing null vectors, respectively, which satisfy the normalization conditions

$k_\mu l^\mu = -1$ and $m_\mu \bar{m}^\mu = 1$. The metric tensor can be represented by $g_{\mu\nu} = -2(l_\mu k_\nu - m_{(\mu} \bar{m}_{\nu)})$, where \bar{m} means the complex-conjugate, and the Greek index runs from 0 to 3.

The directional derivative operators are defined as $D = l^\mu \partial_\mu$, $\Delta = \kappa^\mu \partial_\mu$, and $\delta = m^\mu \partial_\mu$. The spin coefficients are obtained from the projections

$$\begin{aligned}\kappa &= m^\mu l_{\mu;\nu} l^\nu; & \tau &= m^\mu l_{\mu;\nu} k^\nu; & \sigma &= m^\mu l_{\mu;\nu} m^\nu; & \rho &= m^\mu l_{\mu;\nu} \bar{m}^\nu; \\ \hat{\kappa} &= -\bar{m}^\mu k_{\mu;\nu} l^\nu; & \nu &= -\bar{m}^\mu k_{\mu;\nu} k^\nu; & \mu &= -\bar{m}^\mu k_{\mu;\nu} m^\nu; & \lambda &= -\bar{m}^\mu k_{\mu;\nu} \bar{m}^\nu; \\ \epsilon &= \frac{1}{2}(k^\mu l_{\mu;\nu} - m^\mu \bar{m}_{\mu;\nu}) l^\nu; & \gamma &= \frac{1}{2}(k^\mu l_{\mu;\nu} - \bar{m}^\mu m_{\mu;\nu}) k^\nu; \\ \beta &= \frac{1}{2}(k^\mu l_{\mu;\nu} - \bar{m}^\mu m_{\mu;\nu}) m^\nu; & \alpha &= \frac{1}{2}(k^\mu l_{\mu;\nu} - \bar{m}^\mu m_{\mu;\nu}) \bar{m}^\nu,\end{aligned}\quad (1)$$

where “;” stands for covariant derivative. The Weyl scalars related with the curvature Ψ_0 , Ψ_1 , Ψ_2 , Ψ_3 , and Ψ_4 , and the source terms Φ_{ij} , are defined as

$$\begin{aligned}\Psi_0 &= -C_{\mu\nu\lambda\tau} l^\mu m^\nu l^\lambda m^\tau = -C_{lm\bar{m}l}; & \Psi_1 &= -C_{\mu\nu\lambda\tau} l^\mu k^\nu l^\lambda m^\tau = -C_{lk\bar{m}l}; \\ \Psi_2 &= -C_{\mu\nu\lambda\tau} l^\mu m^\nu \bar{m}^\lambda k^\tau = -C_{lm\bar{m}k}; & \Psi_3 &= -C_{\mu\nu\lambda\tau} l^\mu k^\nu \bar{m}^\lambda k^\tau = -C_{lk\bar{m}k}; \\ \Psi_4 &= -C_{\mu\nu\lambda\tau} k^\mu \bar{m}^\nu k^\lambda \bar{m}^\tau = -C_{k\bar{m}k\bar{m}},\end{aligned}\quad (2)$$

and

$$\begin{aligned}\Phi_{00} &= \bar{\Phi}_{00} = 4\pi T_{\mu\nu} l^\mu l^\nu \equiv 4\pi T_{ll}; & \Phi_{01} &= \bar{\Phi}_{10} = 4\pi T_{\mu\nu} l^\mu m^\nu \equiv 4\pi T_{lm}; \\ \Phi_{02} &= \bar{\Phi}_{20} = 4\pi T_{\mu\nu} m^\mu m^\nu \equiv 4\pi T_{mm}; & \Phi_{11} &= 2\pi T_{\mu\nu} (l^\mu k^\nu + m^\mu \bar{m}^\nu) \equiv 2\pi (T_{lk} + T_{m\bar{m}}); \\ \Phi_{22} &= \bar{\Phi}_{22} = 4\pi T_{\mu\nu} k^\mu k^\nu \equiv 4\pi T_{kk}; & \Phi_{12} &= \bar{\Phi}_{21} = 4\pi T_{\mu\nu} k^\mu m^\nu \equiv 4\pi T_{km},\end{aligned}\quad (3)$$

where $C_{\mu\nu\lambda\tau}$ is the Weyl tensor and $T_{\mu\nu}$ is the stress energy tensor of the matter content. The information of the electromagnetic fields is encoded in the scalars,

$$\varphi_0 = F_{\mu\nu} l^\mu m^\nu; \quad \varphi_1 = \frac{1}{2} F_{\mu\nu} (l^\mu k^\nu + \bar{m}^\mu m^\nu); \quad \varphi_2 = F_{\mu\nu} \bar{m}^\mu k^\nu, \quad (4)$$

where $F_{\mu\nu}$ is the Faraday tensor [24]. The stress energy tensor for the electromagnetic field has the form

$$T_{\mu\nu} = \frac{\eta}{4\pi\mu_0} \left(F_{\mu\lambda} F^\lambda_\nu + \frac{1}{4} g_{\mu\nu} F^{\alpha\beta} F_{\alpha\beta} \right), \quad (5)$$

where μ_0 is the magnetic permeability in vacuum. In the rest of this work, we set $\mu_0 = 1$.

From the definition of the scalars Φ_{ij} and Equations (4) and (5), we consider that $\Phi_{ij} = 2\eta \varphi_i \bar{\varphi}_j$. The Latin index runs from 1 to 3. In order to obtain the electromagnetic and gravitational perturbation equations, we depart from the projected Maxwell equations, and the Bianchi identities [21,26] as explained below.

2.1. Maxwell Equations

The dynamics of the electromagnetic fields with sources are described by the Maxwell equations $F^{\mu\nu}_{;\nu} = J^\nu$, where J^ν is the external electric current. The Maxwell equations projected along the tetrad $\eta^{pn} F_{ap|n} = J_a$ (“|” means intrinsic derivative) written in terms of the spin coefficients and the electromagnetic scalars are [27]

$$(D - 2\eta\rho)\varphi_1 - (\bar{\delta} + \eta(\pi - 2\alpha))\varphi_0 + \eta\kappa\varphi_2 = \frac{\eta}{2} J_l, \quad (6)$$

$$(\delta - 2\eta\tau)\varphi_1 - (\Delta + \eta(\mu - 2\gamma))\varphi_0 + \eta\sigma\varphi_2 = \frac{\eta}{2} J_m, \quad (7)$$

$$(D - \eta(\rho - 2\epsilon))\varphi_2 - (\bar{\delta} + 2\eta\pi)\varphi_1 + \eta\lambda\varphi_0 = \frac{\eta}{2} J_{\bar{m}}, \quad (8)$$

$$(\delta - \eta(\tau + 2\beta))\varphi_2 - (\Delta + 2\eta\mu)\varphi_1 + \eta\nu\varphi_0 = \frac{\eta}{2} J_k, \quad (9)$$

where $J_l = J_\mu l^\mu$, $J_m = J_\mu m^\mu$ and $J_k = J_\mu k^\mu$.

2.2. Bianchi Identities

By projecting the equations $R_{\mu\nu}(\lambda\tau;\sigma) = 0$ on the tetrad, one gets a set of equations which include the spin coefficients and operators acting on the Weyl scalars known as the Bianchi identities. In the following, we use two of them: Equations (10) and (11) from [21]:

$$(D + \eta(4\epsilon - \rho))\Psi_4 - (\bar{\delta} + 2\eta(2\pi + \alpha))\Psi_3 + (3\eta\Psi_2 + 2\Phi_{11})\lambda \\ = \eta(\bar{\delta} + 2\eta(\alpha - \bar{\tau}))\Phi_{21} - \eta(\Delta + \eta(\bar{\mu} + 2\gamma - 2\bar{\gamma}))\Phi_{20} + 2\nu\Phi_{10} + \bar{\sigma}\Phi_{22}, \quad (10)$$

$$-(\delta + \eta(4\beta - \tau))\Psi_4 + (\Delta + 2\eta(\gamma + 2\mu))\Psi_3 - (3\eta\Psi_2 - 2\Phi_{11})\nu \\ = \eta(\Delta + 2\eta(\bar{\mu} + \gamma))\Phi_{21} - \eta(\bar{\delta} + \eta(-\bar{\tau} + 2\alpha + 2\bar{\beta}))\Phi_{22} - \bar{\nu}\Phi_{20} + 2\lambda\Phi_{12}. \quad (11)$$

For a detailed description on the projections, we refer the reader to [26]. Similarly, projecting the Weyl tensor on the tetrad, we obtain the following expression,

$$\Psi_4 + (\Delta + \eta(\mu + \bar{\mu} + 3\gamma - \bar{\gamma}))\lambda - (\bar{\delta} + \eta(3\alpha + \bar{\beta} + \pi - \bar{\tau}))\nu = 0. \quad (12)$$

The equations above are the three equations needed in the forthcoming derivation.

3. Equations for the Perturbations

Corresponding to the six parameters of the Lorentz group of transformation, there are six degrees of freedom to rotate a chosen tetrad frame. It is usual to encode a general Lorentz transformation in terms of the basis vectors l , k , and m , and classify them in three classes of rotations, each one leaving an invariant vector under such transformation. The effect of the basis transformation on the various Newman–Penrose quantities can be found in [21]. For instance, an infinitesimal rotation of class I will change the Weyl and the Maxwell scalars to first order in a , according to the following scheme:

$$\Psi_0 \rightarrow \Psi_0, \quad \Psi_1 \rightarrow \Psi_1 + a\Psi_0^{(1)}, \quad \Psi_2 \rightarrow \Psi_2 + 2a\Psi_1^{(1)}, \\ \Psi_3 \rightarrow \Psi_3 + 3a\Psi_2^{(1)}, \quad \Psi_4 \rightarrow \Psi_4 + 4a\Psi_3^{(1)}, \quad (13)$$

and

$$\varphi_0 \rightarrow \varphi_0, \quad \varphi_1 \rightarrow \varphi_1 + a\varphi_0^{(1)}, \quad \varphi_2 \rightarrow \varphi_2 + 2a\varphi_1^{(1)}, \quad (14)$$

where the superscript denotes a perturbed quantity. For the Reissner–Nordström spacetime, the background scalars Ψ_2 and φ_1 are the only ones that are nonzero. Consequently, Ψ_0 , Ψ_1 , Ψ_2 , Ψ_4 , φ_0 , and φ_1 are unaffected to first order under an infinitesimal rotation. However, Ψ_3 and φ_2 are indeed affected since Ψ_2 and φ_1 are different from zero in the background. Nevertheless, the radiative combination

$$\chi = \eta(2\varphi_1\Psi_3^{(1)} - 3\Psi_2\varphi_2^{(1)}), \quad (15)$$

is invariant to first order and, as it is shown below, one can get a coupled system of equations for this function, χ , and the perturbed Weyl component $\Psi_4^{(1)}$. In the following, we describe the general procedure to find such a coupled set of equations in detail.

3.1. Perturbed Maxwell Equations

The ingoing and outgoing electromagnetic radiation is given by the perturbations of the scalars φ_0 and φ_2 , respectively. However, in a charged spacetime, the outgoing electromagnetic perturbations couple with the perturbations of the Weyl scalar Ψ_3 , which carries the so-called electromagnetic part of the spacetime. A similar coupling occurs with the ingoing perturbation φ_0 , and Ψ_1 . In this section, we derive an equation for the perturbation of φ_2 .

First, consider the following identities relating the derivative operators [21]:

$$[\Delta, \bar{\delta}] = \eta \nu D + \eta(\alpha + \bar{\beta} - \bar{\tau})\Delta - \eta(\bar{\mu} - \bar{\gamma} + \gamma)\bar{\delta} - \eta\lambda\delta, \quad (16)$$

$$\eta\Delta\pi = \eta D\nu - \mu(\pi + \bar{\tau}) - \lambda(\bar{\pi} + \tau) - \pi(\gamma - \bar{\gamma}) + \nu(3\epsilon + \bar{\epsilon}) - \eta\Psi_3 - \Phi_{21}, \quad (17)$$

$$\eta\bar{\delta}\mu = \eta\delta\lambda - \nu(\rho - \bar{\rho}) - \pi(\mu - \bar{\mu}) - \mu(\alpha + \bar{\beta}) - \lambda(\bar{\alpha} - 3\beta) + \eta\Psi_3 - \Phi_{21}. \quad (18)$$

Second, on the projected Maxwell's Equations (6)–(9), operate $[\Delta + \eta(\bar{\mu} - \bar{\gamma} + \gamma + 2\mu)]$ on Equation (8) and $[\bar{\delta} - \eta(\bar{\tau} - \alpha - \bar{\beta} - 2\pi)]$ on Equation (9); then, subtract these equations and, after some algebra, one arrives to the following expression:

$$\begin{aligned} & [(\Delta + \eta(\gamma - \bar{\gamma} + 2\mu + \bar{\mu}))(D - \eta(\rho_s + 2\epsilon)) - (\bar{\delta} + \eta(\alpha + \bar{\beta} + 2\pi_s - \bar{\tau}))(\delta - \eta(\tau + 2\beta))] \varphi_2 + \\ & - [\eta\nu D \varphi_1 - \eta\lambda\delta \varphi_1 + 2\varphi_1 ((D + \eta(3\epsilon + \bar{\epsilon} + \rho - \bar{\rho}_s))\eta\nu - (\delta + \eta(\bar{\pi} + \tau - \bar{\alpha} + 3\beta))\eta\lambda - 4\eta\Psi_3)] + \\ & \varphi_0 [(\Delta + \eta(\bar{\mu} - \bar{\gamma} + \gamma + 2\mu))\eta\lambda - (\bar{\delta} - \eta(\bar{\tau} - \alpha - \bar{\beta} - 2\pi))\eta\nu] + \eta\lambda \Delta \varphi_0 - \eta\nu \bar{\delta} \varphi_0 = \frac{\eta}{2} J_2, \end{aligned} \quad (19)$$

where

$$J_2 = (\Delta + \eta(\gamma - \bar{\gamma} + 2\mu + \bar{\mu}))J_{\bar{m}} - (\bar{\delta} + \eta(\alpha + \bar{\beta} + 2\pi - \bar{\tau}))J_k. \quad (20)$$

In order to describe the perturbation of φ_2 , let us perform a first-order perturbation of the form $f \rightarrow f + f^{(1)}$ in all the functions on Equation (19). In a Reissner–Nordström-like background, considering that the spacetimes are type D , in Petrov classification, the spinor quantities $\nu, \lambda, \kappa, \sigma$ are zero. Furthermore, we consider spherical symmetry and that φ_0, φ_2 are also zero. These considerations imply

$$\begin{aligned} & [(\Delta + \eta(\gamma - \bar{\gamma} + 2\mu + \bar{\mu}))(D - \eta(\rho + 2\epsilon)) - (\bar{\delta} + \eta(\alpha + \bar{\beta} + 2\pi - \bar{\tau}))(\delta - \eta(\tau + 2\beta))] \varphi_2^{(1)} + \\ & - \left\{ \eta\nu^{(1)} D \varphi_1 - \eta\lambda^{(1)} \delta \varphi_1 + 2\varphi_1 \left[(D + \eta(3\epsilon + \bar{\epsilon} + \rho - \bar{\rho}))\eta\nu^{(1)} - (\delta + \eta(\bar{\pi} + \tau - \bar{\alpha} + 3\beta))\eta\lambda^{(1)} - \right. \right. \\ & \left. \left. 4\eta\Psi_3^{(1)} \right] \right\} = \frac{\eta}{2} J_2^{(1)}, \end{aligned} \quad (21)$$

with the perturbed current term defined as

$$J_2^{(1)} = (\Delta + \eta(\gamma - \bar{\gamma} + 2\mu + \bar{\mu}))J_{\bar{m}}^{(1)} - (\bar{\delta} + \eta(\alpha + \bar{\beta} + 2\pi - \bar{\tau}))J_k^{(1)}. \quad (22)$$

In previous equation, we only kept first-order and nonvanishing background quantities. Equation (21) can be further simplified using background Maxwell equations Equation (6): $D\varphi_1 = 2\eta\rho\varphi_1$; and Equation (7): $\delta\varphi_1 = 2\eta\tau\varphi_1$, in order to obtain

$$\begin{aligned} & [(\Delta + \eta(\bar{\mu} - \bar{\gamma} + \gamma + 2\mu))(D - \eta(\rho - 2\epsilon)) - (\bar{\delta} - \eta(\bar{\tau} - \alpha - \bar{\beta} - 2\pi))(\delta - \eta(\tau - 2\beta))] \varphi_2^{(1)} = \\ & 2\eta\varphi_1 [(D + \eta(3\epsilon + \bar{\epsilon} + 2\rho - \bar{\rho}))\nu^{(1)} - (\delta + \eta(\bar{\pi} - \bar{\alpha} + 3\beta + 2\tau))\lambda^{(1)} - 2\Psi_3^{(1)}] + \frac{\eta}{2} J_2^{(1)}. \end{aligned} \quad (23)$$

This equation can be written in a more convenient form using the commutation expressions for the operators acting on $\varphi_2^{(1)}$,

$$[\Delta, D] = \eta(\gamma + \bar{\gamma})D + \eta(\epsilon + \bar{\epsilon})\Delta - \eta(\bar{\tau} + \pi)\delta - \eta(\tau + \bar{\pi})\bar{\delta}, \quad (24)$$

$$[\bar{\delta}, \delta] = \eta(\bar{\mu} - \mu)D + \eta(\bar{\rho} - \rho)\Delta + \eta(\alpha - \bar{\beta})\delta + \eta(\beta - \bar{\alpha})\bar{\delta}, \quad (25)$$

and the Ricci identities [28]:

$$\eta D\mu = \eta\delta\pi + (\bar{\rho}\mu + \sigma\lambda) + \pi(\bar{\pi} - \bar{\alpha} + \beta) - \mu(\epsilon + \bar{\epsilon}) - \nu\kappa + \eta\Psi_2 + 2R, \quad (26)$$

$$\eta\bar{\delta}\beta = -\eta\delta\alpha - (\mu\rho - \lambda\sigma) - \alpha\bar{\alpha} - \beta(\bar{\beta} - 2\alpha) - \gamma(\rho - \bar{\rho}) - \epsilon(\mu - \bar{\mu}_s) + \eta\Psi_2 - \Phi_{11} - R, \quad (27)$$

in Equation (23). After some algebra, one arrives to the following expression relating the perturbations $\varphi_2^{(1)}, \nu^{(1)}, \lambda^{(1)}, \Psi_3^{(1)}$ with the current $J_2^{(1)}$

$$[(D - \eta(\rho - 3\epsilon - \bar{\epsilon}))(\Delta + \eta(2\mu + \bar{\mu} + 2\gamma)) - (\delta + \eta(3\beta - \bar{\alpha} - \tau))(\bar{\delta} + \eta(2\alpha + 2\pi - \tau)) - 3\eta\Psi_2]\varphi_2^{(1)} = 2\eta\varphi_1[(D + \eta(2\rho - \bar{\rho} + 3\epsilon + \bar{\epsilon}))v^{(1)} - (\delta - \eta(\bar{\alpha} - 3\beta - \bar{\pi} - 2\tau_s))\lambda^{(1)} - 2\Psi_3^{(1)}] + \frac{\eta}{2}J_2^{(1)}. \quad (28)$$

In order to simplify the notation, let us define the derivative operators as follows:

$$D_N = D + \eta N \rho + 4\eta \epsilon, \quad \Delta_N = \Delta + \eta N \mu, \quad (29)$$

$$\delta_N = \delta + \eta N \beta, \quad \bar{\delta}_N = \bar{\delta} + \eta N \bar{\beta}, \quad (30)$$

where N takes integer values. In terms of these expressions, we can rewrite the electromagnetic perturbation Equation (28) as

$$D_{+1}v^{(1)} - \delta_4\lambda^{(1)} = \frac{1}{2\eta\varphi_1} [D_{-1}\Delta_3 - \delta_4\bar{\delta}_{-2} - 6\eta\Psi_2]\varphi_2^{(1)} + 2\Psi_3^{(1)} + \frac{1}{4\varphi_1}J_2^{(1)}. \quad (31)$$

With the notation of Equations (29) and (30), we denoted $D_{+1} = D + \eta((+1)\rho + 4\epsilon)$ for $N = +1$, $D_{-1} = D + \eta((-1)\rho + 4\epsilon)$ for $N = -1$, and analogously for other values and operators.

3.2. Perturbed Bianchi Identities

In order to derive the equations for the perturbations $\Psi_4^{(1)}$ and $\Psi_3^{(1)}$, we start by perturbing the the Bianchi identities Equations (10) and (11). As mentioned above, in a Petrov-type D space-time background, the background Weyl scalars vanish except Ψ_2 , and the spinor coefficients $\kappa, \nu, \lambda, \sigma$ are equal to zero. Furthermore, we consider spacetimes, such as the Schwarzschild or the Reissner–Nordstrom, in which it is always possible to choose a tetrad so that the nonzero spinor coefficients are real. Finally, we consider stress energy tensors of the same form as the Reissner–Nordström one, so that the only nonzero Ricci scalar is Φ_{11} .

By performing a first-order perturbation in Equations (10) and (11), one gets the following two equations:

$$D_{-1}\Psi_4^{(1)} - \bar{\delta}_{-2}\Psi_3^{(1)} + (3\eta\Psi_2 + 2\Phi_{11})\lambda^{(1)} - \varphi_1\bar{\delta}_{-2}\varphi_2^{(1)} = 0, \quad (32)$$

$$-\delta_4\Psi_4^{(1)} + \Delta_4\Psi_3^{(1)} - (3\eta\Psi_2 - 2\Phi_{11})\nu^{(1)} - \Delta_2\varphi_1\varphi_2^{(1)} = -\eta 4\pi\bar{\delta}_0T_{22}^{(1)}. \quad (33)$$

In the forthcoming analysis, we consider that the external matter that causes the perturbation satisfies $T_{\mu\nu}^{(1)}k^\mu k^\nu \equiv T_{22}^{(1)} = \Phi_{22}^{(1)}/4\pi \neq 0$. As we show below, this condition is consistent with matter falling in the radial direction only.

In an analogous manner, the perturbation of Equation (12) gives

$$\Psi_4^{(1)} + \Delta_2\lambda^{(1)} - \bar{\delta}_{-2}\nu^{(1)} = 0. \quad (34)$$

Finally, the following Ricci identities describing the action of the operators D and Δ on the unperturbed fields are also useful:

$$D\Psi_2 = \rho(3\eta\Psi_2 + 2\Phi_{11}), \quad D\Phi_{11} = 4\eta\rho\Phi_{11}, \quad (35)$$

$$\Delta\Psi_2 = -\mu(3\eta\Psi_2 + 2\Phi_{11}), \quad \Delta\Phi_{11} = -4\eta\mu\Phi_{11}, \quad (36)$$

$$D\varphi_1 = 2\eta\rho\varphi_1, \quad \Delta\varphi_1 = -2\eta\mu\varphi_1. \quad (37)$$

3.3. System of Equations for the Perturbations

In the previous subsection, we obtained four equations relating the perturbations $\Psi_4^{(1)}, \Psi_3^{(1)}, \varphi_2^{(1)}, \nu^{(1)}$, and $\lambda^{(1)}$ due to the perturbed sources $T_{22}^{(1)}$ and $J_2^{(1)}$. It is an underdetermined system, with four equations for five unknowns. However, as we show, one can

partially solve such a system using the particular combination of $\Psi_3^{(1)}$ and $\varphi_2^{(1)}$ given by Equation (15) and obtain a subsystem of coupled equations for $\Psi_4^{(1)}$ and χ . This remarkable combination has been related to a freedom in the rotation of the tetrad [21], although its physical meaning is not clearly understood and, to our knowledge, the physical meaning of such a combination has not been discussed in the literature. In this section, we present a detailed derivation of such a subsystem of equations.

Acting with Δ_5 on Equation (32), and with $\bar{\delta}_{-2}$ on Equation (33), adding and using the identity $\Delta_{1-q} \bar{\delta}_p \equiv \bar{\delta}_p \Delta_{-q}$ with $q = -4$, $p = -2$, one gets

$$\begin{aligned} & (\Delta_5 D_{-1} - \bar{\delta}_{-2} \delta_4) \Psi_4^{(1)} + ((3\eta \Psi_2 + 2\Phi_{11}) \Delta_5 + 3\eta (\Delta \Psi_2) + 2(\Delta \Phi_{11})) \lambda^{(1)} \\ & - (3\eta \Psi_2 - 2\Phi_{11}) \bar{\delta}_{-2} \nu^{(1)} - \varphi_1 \left(\left(\Delta_5 + \frac{\Delta \varphi_1}{\varphi_1} \right) \bar{\delta}_{-2} + \bar{\delta}_{-2} \left(\Delta_2 + \frac{\Delta \varphi_1}{\varphi_1} \right) \right) \varphi_2^{(1)} = -4\pi \eta \bar{\delta}_{-2} \bar{\delta}_0 T_{22}^{(1)}. \end{aligned} \quad (38)$$

A further simplification can be done using the fact that the action of the operator Δ_N on φ_1 is

$$\Delta_N (\varphi_1 f) \equiv \varphi_1 \Delta_{N-2} f, \quad (39)$$

for an arbitrary function f . Thus, using Equations (38) and (39), it takes the form

$$\begin{aligned} & (\Delta_5 D_{-1} - \bar{\delta}_{-2} \delta_4) \Psi_4^{(1)} + 3\eta \Psi_2 \left((\Delta_2 + 3\eta \mu) \lambda^{(1)} - \bar{\delta}_{-2} \nu^{(1)} \right) + 2\Phi_{11} \left((\Delta_2 + 3\eta \mu) \lambda^{(1)} + \bar{\delta}_{-2} \nu^{(1)} \right) \\ & + (3\eta (-\mu) (3\eta \Psi_2 + 2\Phi_{11}) + 2(-4\eta \mu \Phi_{11})) \lambda^{(1)} - \varphi_1 (\Delta_3 \bar{\delta}_{-2} + \bar{\delta}_{-2} \Delta_0) \varphi_2^{(1)} = -4\pi \eta \bar{\delta}_{-2} \bar{\delta}_0 T_{22}^{(1)}. \end{aligned} \quad (40)$$

Substituting in the previous equation $\Delta_2 \lambda^{(1)}$ given by Equation (34) and after collecting terms, one gets

$$\begin{aligned} & (\Delta_5 D_{-1} - \bar{\delta}_{-2} \delta_4 - 3\eta \Psi_2 - 2\Phi_{11}) \Psi_4^{(1)} + 4\Phi_{11} \bar{\delta}_{-2} \nu^{(1)} \\ & + \mu (9\Psi_2 + 6\eta \Phi_{11} - 9\Psi_2 - 6\eta \Phi_{11} - 8\eta \Phi_{11}) \lambda^{(1)} - \varphi_1 (\Delta_3 + \Delta_1) \bar{\delta}_{-2} \varphi_2^{(1)} = -4\pi \eta \bar{\delta}_{-2} \bar{\delta}_0 T_{22}^{(1)}. \end{aligned} \quad (41)$$

Simplifying and using the definition of the operator Δ_N in Equation (29), one obtains

$$\begin{aligned} & (\Delta_5 D_{-1} - \bar{\delta}_{-2} \delta_4 - 3\eta \Psi_2 - 2\Phi_{11}) \Psi_4^{(1)} - 2\varphi_1 \Delta_2 \bar{\delta}_{-2} \varphi_2^{(1)} - 8\eta \mu \Phi_{11} \lambda^{(1)} + 4\Phi_{11} \bar{\delta}_{-2} \nu^{(1)} \\ & = -4\pi \eta \bar{\delta}_{-2} \bar{\delta}_0 T_{22}^{(1)}. \end{aligned} \quad (42)$$

Next, one can use Equations (32) and (33) to express the perturbed spinors as

$$\lambda^{(1)} = \frac{1}{2\Phi_{11} + 3\eta \Psi_2} \left(-D_{-1} \Psi_4^{(1)} + \bar{\delta}_{-2} \Psi_3^{(1)} + \varphi_1 \bar{\delta}_{-2} \varphi_2^{(1)} \right), \quad (43)$$

$$\nu^{(1)} = \frac{1}{2\Phi_{11} - 3\eta \Psi_2} \left(\delta_4 \Psi_4^{(1)} - \Delta_4 \Psi_3^{(1)} + \Delta_2 \varphi_1 \varphi_2^{(1)} - 4\pi \eta \bar{\delta}_0 T_{22}^{(1)} \right), \quad (44)$$

and substitute them in Equation (42). The resulting equation is

$$\mathbf{A}_1 \Psi_4^{(1)} + \mathbf{A}_2 \varphi_2^{(1)} + \mathbf{A}_3 \Psi_3^{(1)} = 4\pi \mathbf{A}_4 T_{22}^{(1)}, \quad (45)$$

where we have defined the operators

$$\mathbf{A}_1 = \left(\Delta + 3\eta \mu \frac{6\Phi_{11} + 5\eta \Psi_2}{(2\Phi_{11} + 3\eta \Psi_2)^2} \right) D_{-1} - 2\Phi_{11} - 3\eta \Psi_2 + \frac{2\Phi_{11} + 3\eta \Psi_2}{2\Phi_{11} - 3\eta \Psi_2} \bar{\delta}_{-2} \delta_4, \quad (46)$$

$$\mathbf{A}_2 = \frac{(2\varphi_1)(3\eta \Psi_2)}{2\Phi_{11} - 3\eta \Psi_2} \left(\Delta - \frac{2\eta \mu (2\Phi_{11}^2 - 3\Psi_2^2 - 3\Phi_{11} \eta \Psi_2)}{\eta \Psi_2 (2\Phi_{11} + 3\eta \Psi_2)} \right) \bar{\delta}_{-2}, \quad (47)$$

$$\mathbf{A}_3 = -\frac{4\Phi_{11}}{2\Phi_{11} - 3\eta \Psi_2} \left(\Delta + \eta \mu \frac{14\Phi_{11} + 9\eta \Psi_2}{2\Phi_{11} + 3\eta \Psi_2} \right) \bar{\delta}_{-2}, \quad (48)$$

$$\mathbf{A}_4 = \eta \frac{2\Phi_{11} + 3\eta \Psi_2}{2\Phi_{11} - 3\eta \Psi_2} \bar{\delta}_{-2} \bar{\delta}_0. \quad (49)$$

Remarkably, the operator \mathbf{A}_2 acting on $-3\eta\Psi_2\varphi_2^{(1)}$ takes the same form as the operator \mathbf{A}_3 acting on $2\eta\varphi_1\Psi_3^{(1)}$. Indeed, one can write the action of \mathbf{A}_2 and \mathbf{A}_3 on $\varphi_2^{(1)}$ and $\Psi_3^{(1)}$ as

$$\begin{aligned}\mathbf{A}_2\varphi_2^{(1)} &= \mathbf{A}_2\left(\frac{1}{(-3\eta\Psi_2)}(-3\eta\Psi_2\varphi_2^{(1)})\right) \\ &= -\frac{(2\varphi_1)}{(2\Phi_{11}-3\eta\Psi_2)}\left(\Delta+3\eta\mu\frac{6\Phi_{11}+5\eta\Psi_2}{2\Phi_{11}+3\eta\Psi_2}\right)\bar{\delta}_{-2}(-3\eta\Psi_2\varphi_2^{(1)}),\end{aligned}\quad (50)$$

$$\begin{aligned}\mathbf{A}_3\Psi_3^{(1)} &= \mathbf{A}_{3a}\left(\frac{1}{2\eta\varphi_1}(2\eta\varphi_1\Psi_3^{(1)})\right) \\ &= -\frac{(2\varphi_1)}{(2\Phi_{11}-3\eta\Psi_2)}\left(\Delta+3\eta\mu\frac{6\Phi_{11}+5\eta\Psi_2}{2\Phi_{11}+3\eta\Psi_2}\right)\bar{\delta}_{-2}(2\eta\varphi_1\Psi_3^{(1)}),\end{aligned}\quad (51)$$

and using this remarkable property in Equation (45), one gets

$$\mathbf{A}_1\Psi_4^{(1)}+\mathbf{A}_5\chi=4\pi\mathbf{A}_4T_{22}^{(1)},\quad (52)$$

where

$$\chi=\eta\left(2\varphi_1\Psi_3^{(1)}-3\Psi_2\varphi_2^{(1)}\right),\quad (53)$$

and

$$\mathbf{A}_5=-\frac{(2\varphi_1)}{(2\Phi_{11}-3\eta\Psi_2)}\left(\Delta+3\eta\mu\frac{6\Phi_{11}+5\eta\Psi_2}{2\Phi_{11}+3\eta\Psi_2}\right)\bar{\delta}_{-2}.\quad (54)$$

In order to derive a second equation for $\Psi_4^{(1)}$ and χ , we first apply the operator δ_4 on Equation (32), and the operator D_{-2} on Equation (33), and adding up and using the rule of commutation $\delta_4 D_{-1}-D_{-2}\delta_4=0$, we eliminate $\Psi_4^{(1)}$ in the equation. The resulting equation is

$$\begin{aligned}&(D_{-2}\Delta_4-\delta_4\bar{\delta}_{-2})\Psi_3^{(1)}+(3\eta\Psi_2+2\Phi_{11})\delta_4\lambda^{(1)}+(-\varphi_1\delta_4\bar{\delta}_{-2}-D_{-2}\Delta_2\varphi_1)\varphi_2^{(1)}- \\ &[(3\eta\Psi_2-2\Phi_{11})D_{-2}+3\eta(D\Psi_2)-2(D\Phi_{11})]\nu^{(1)}=-4\pi\eta D_{-2}\bar{\delta}_0T_{22}^{(1)}.\end{aligned}\quad (55)$$

Considering that $D_{-2}=D_{+1}-3\eta\rho_s$, we can use the Maxwell Equation (31) to expand this last equation as follows:

$$\begin{aligned}&(D_{-2}\Delta_4-\delta_4\bar{\delta}_{-2})\Psi_3^{(1)}-\varphi_1((D_{-2}+2\eta\rho)(\Delta_2-2\eta\mu)+\delta_4\bar{\delta}_{-2})\varphi_2^{(1)} \\ &-3\eta\Psi_2\left(\frac{1}{2\eta\varphi_1}[D_{-1}\Delta_3-\delta_4\bar{\delta}_{-2}-6\eta\Psi_2]\varphi_2^{(1)}+2\Psi_3^{(1)}+\frac{1}{4\varphi_1}J_2^{(1)}\right)+ \\ &2\Phi_{11}\left(2\delta_4\lambda^{(1)}+\frac{1}{2\eta\varphi_1}[D_{-1}\Delta_3-\delta_4\bar{\delta}_{-2}-6\eta\Psi_2]\varphi_2^{(1)}+2\Psi_3^{(1)}+\frac{1}{4\varphi_1}J_2^{(1)}\right)-4\eta\rho\Phi_{11}\nu^{(1)} \\ &=-4\pi\eta D_{-2}\bar{\delta}_0T_{22}^{(1)}.\end{aligned}\quad (56)$$

After some algebra, the previous equation becomes

$$\begin{aligned}&(D_{-2}\Delta_4-\delta_4\bar{\delta}_{-2}-2(3\eta\Psi_2-2\Phi_{11}))\Psi_3^{(1)}+ \\ &-\left(\varphi_1(D_0\Delta_0+\delta_4\bar{\delta}_{-2})+\frac{3\eta\Psi_2-2\Phi_{11}}{2\eta\varphi_1}(D_{-1}\Delta_3-\delta_4\bar{\delta}_{-2}-6\eta\Psi_2)\right)\varphi_2^{(1)}+ \\ &4\Phi_{11}\delta_4\lambda^{(1)}-4\eta\rho\Phi_{11}\nu^{(1)}=-4\pi\eta D_{-2}\bar{\delta}_0T_{22}^{(1)}+\frac{1}{4\varphi_1}(3\eta\Psi_2-2\Phi_{11})J_2^{(1)}.\end{aligned}\quad (57)$$

From Equations (32) and (33), one can obtain the following expressions for the terms involving the perturbed spinor coefficients:

$$4 \Phi_{11} \delta_4 \lambda^{(1)} = \frac{4 \Phi_{11}}{2 \Phi_{11} + 3 \eta \Psi_2} \left(-D_{-2} \delta_4 \Psi_4^{(1)} + \delta_4 \bar{\delta}_{-2} \Psi_3^{(1)} + \varphi_1 \delta_4 \bar{\delta}_{-2} \varphi_2^{(1)} \right), \quad (58)$$

$$-4 \eta \rho \Phi_{11} \nu^{(1)} = -\eta \rho \frac{4 \Phi_{11}}{2 \Phi_{11} - 3 \eta \Psi_2} \left(\delta_4 \Psi_4^{(1)} - \Delta_4 \Psi_3^{(1)} + \varphi_1 \Delta_0 \varphi_2^{(1)} - 4 \pi \eta \bar{\delta}_0 T_{22}^{(1)} \right). \quad (59)$$

Using these expressions in Equation (57), and after some simplifications, we obtain an equation involving only $\Psi_4^{(1)}, \Psi_3^{(1)}, \varphi_2^{(1)}$, and the sources

$$\mathbf{B}_1 \Psi_4^{(1)} + (\mathbf{O}_{1r} + \mathbf{O}_{1a}) \Psi_3^{(1)} + (\mathbf{O}_{2r} + \mathbf{O}_{2a}) \varphi_2^{(1)} = 4 \pi \mathbf{B}_3 T_{22}^{(1)} + \frac{1}{4 \varphi_1} (3 \eta \Psi_2 - 2 \Phi_{11}) J_2^{(1)}, \quad (60)$$

where the operators have the form

$$\mathbf{B}_1 = -\frac{4 \Phi_{11}}{2 \Phi_{11} + 3 \eta \Psi_2} \left(D_0 - \eta \rho \frac{2 \Phi_{11} - 9 \eta \Psi_2}{2 \Phi_{11} - 3 \eta \Psi_2} \right) \delta_4, \quad (61)$$

$$\mathbf{O}_{1r} = \left(D_0 + 6 \eta \rho \frac{\eta \Psi_2}{2 \Phi_{11} - 3 \eta \Psi_2} \right) (\Delta_0 + 4 \eta \mu), \quad (62)$$

$$\mathbf{O}_{1a} = \frac{2 \Phi_{11} - 3 \eta \Psi_2}{2 \Phi_{11} + 3 \eta \Psi_2} [\delta_4 \bar{\delta}_{-2} + 2 (2 \Phi_{11} + 3 \eta \Psi_2)], \quad (63)$$

$$\mathbf{O}_{2r} = -\varphi_1 \left(D_0 + 4 \eta \rho \frac{\Phi_{11}}{2 \Phi_{11} - 3 \eta \Psi_2} \right) \Delta_0 + \frac{2 \Phi_{11} - 3 \eta \Psi_2}{2 \eta \varphi_1} (D_0 - \eta \rho) (\Delta_0 + 3 \eta \mu), \quad (64)$$

$$\mathbf{O}_{2a} = -\left(\frac{3 \eta \Psi_2}{2 \eta \varphi_1} \right) \frac{2 \Phi_{11} - 3 \eta \Psi_2}{2 \Phi_{11} + 3 \eta \Psi_2} [\delta_4 \bar{\delta}_{-2} + 2 (2 \Phi_{11} + 3 \eta \Psi_2)], \quad (65)$$

$$\mathbf{B}_3 = -\eta \left(D_0 + 6 \eta \rho \frac{\eta \Psi_2}{2 \Phi_{11} - 3 \eta \Psi_2} \right) \bar{\delta}_0. \quad (66)$$

We have collected the operators acting on $\Psi_3^{(1)}$ and on $\varphi_2^{(1)}$ in those involving D_N and Δ_N , and the rest, as long as they involve more algebraic manipulation in the next steps in the derivation. Indeed, using $\Psi_3^{(1)} = (2 \eta \varphi_1 / 2 \eta \varphi_1) \Psi_3^{(1)}$ and $\varphi_2^{(1)} = (-3 \eta \Psi_2 / -3 \eta \Psi_2) \varphi_2^{(1)}$ in $\mathbf{O}_{1a} \Psi_3^{(1)}$, one gets

$$\mathbf{O}_{1a} \Psi_3^{(1)} = \mathbf{O}_{1a} \left(\frac{2 \eta \varphi_1}{2 \eta \varphi_1} \Psi_3^{(1)} \right) = \mathcal{O}_{1a} (2 \eta \varphi_1 \Psi_3^{(1)}), \quad (67)$$

and for the radial operators

$$\mathbf{O}_{1r} \Psi_3^{(1)} = \mathbf{O}_{1r} \frac{1}{(2 \eta \varphi_1)} (2 \eta \varphi_1 \Psi_3^{(1)}) = \mathcal{O}_{1r} (2 \eta \varphi_1 \Psi_3^{(1)}), \quad (68)$$

$$\mathbf{O}_{2r} \varphi_2^{(1)} = \mathbf{O}_{2r} \frac{1}{(-3 \eta \Psi_2)} (-3 \eta \Psi_2 \varphi_2^{(1)}) = \mathcal{O}_{2r} (-3 \eta \Psi_2 \varphi_2^{(1)}), \quad (69)$$

where \mathcal{O}_{1a} , \mathcal{O}_{1r} , and \mathcal{O}_{2r} have the form

$$\mathcal{O}_{1a} = \left(\frac{1}{2 \eta \varphi_1} \right) \frac{2 \Phi_{11} - 3 \eta \Psi_2}{2 \Phi_{11} + 3 \eta \Psi_2} (\delta_4 \bar{\delta}_{-2} + 2 (3 \eta \Psi_2 + 2 \Phi_{11})), \quad (70)$$

and

$$\mathcal{O}_{1r} = \frac{1}{2\eta\varphi_1} \left(D_{-2} - \frac{D\varphi_1}{\varphi_1} + 4\eta\rho \frac{\Phi_{11}}{2\Phi_{11} - 3\eta\Psi_2} \right) \left(\Delta_4 - \frac{\Delta\varphi_1}{\varphi_1} \right) \quad (71)$$

$$\begin{aligned} &= \frac{1}{2\eta\varphi_1} \left(D_0\Delta_0 + 6\eta\mu D_0 - 4\eta\rho \frac{\Phi_{11} - 3\eta\Psi_2}{2\Phi_{11} - 3\eta\Psi_2} \Delta_0 + 6\eta(D\mu) - 24\rho\mu \frac{\Phi_{11} - 3\eta\Psi_2}{2\Phi_{11} - 3\eta\Psi_2} \right), \\ \mathcal{O}_{2r} &= -\frac{1}{3\eta\Psi_2} \left[-\varphi_1 \left(D_0 - \frac{D\Psi_2}{\Psi_2} + 4\eta\rho \frac{\Phi_{11}}{2\Phi_{11} - 3\eta\Psi_2} \right) \left(\Delta_0 - \frac{\Delta\Psi_2}{\Psi_2} \right) \right. \\ &\quad \left. + \frac{2\Phi_{11} - 3\eta\Psi_2}{2\eta\varphi_1} \left(D_{-1} - \frac{D\Psi_2}{\Psi_2} \right) \left(\Delta_3 - \frac{\Delta\Psi_2}{\Psi_2} \right) \right] \\ &= a_0 D_0 \Delta_0 + a_1 D_0 + a_2 \Delta_0 + a_3 (D\mu) + a_4. \end{aligned} \quad (72)$$

We have written the operator \mathcal{O}_{2r} as a sum of different operators, since, as shown below, each element of the sum is equal to the corresponding element of \mathcal{O}_{1r} :

$$a_0 = -\frac{1}{3\eta\Psi_2} \left(-\varphi_1 + \frac{2\Phi_{11} - 3\eta\Psi_2}{2\eta\varphi_1} \right) = \frac{1}{2\eta\varphi_1}, \quad (73)$$

$$a_1 = -\frac{1}{3\eta\Psi_2} \frac{\eta\mu}{2\eta\varphi_1\eta\Psi_2} (-2\Phi_{11}(2\Phi_{11} + 3\eta\Psi_2) + 2(2\Phi_{11} - 3\eta\Psi_2)(\Phi_{11} + 3\eta\Psi_2)) = \frac{1}{2\eta\varphi_1} 6\eta\mu, \quad (74)$$

$$\begin{aligned} a_2 &= -\frac{1}{3\eta\Psi_2} \frac{-\eta\rho}{2\eta\varphi_1\eta\Psi_2(2\Phi_{11} - 3\eta\Psi_2)} (-2\Phi_{11}(4\Phi_{11}^2 - 4\eta\Psi_2\Phi_{11} - 9\eta\Psi_2^2) \\ &\quad + 2(2\Phi_{11} - 3\eta\Psi_2)^2(\Phi_{11} + 2\eta\Psi_2)) = \frac{1}{2\eta\varphi_1} \left(-4\eta\rho \frac{\Phi_{11} - 3\eta\Psi_2}{2\Phi_{11} - 3\eta\Psi_2} \right), \end{aligned} \quad (75)$$

$$a_3 = -\frac{1}{3\eta\Psi_2} \frac{\eta}{\eta\Psi_2 2\eta\varphi_1} (-2\Phi_{11}(2\Phi_{11} + 3\eta\Psi_2) + (2\Phi_{11} - 3\eta\Psi_2)(\Phi_{11} + 3\eta\Psi_2)) = \frac{1}{2\eta\varphi_1} 6\eta, \quad (76)$$

$$\begin{aligned} a_4 &= -\frac{1}{3\eta\Psi_2} \left(-\varphi_1 \left(2\mu \left(D \frac{\Phi_{11}}{\Psi_2} \right) - \rho\mu \frac{(4\Phi_{11}^2 - 4\eta\Psi_2\Phi_{11} - 9\eta\Psi_2^2)(2\Phi_{11} + 3\eta\Psi_2)}{\Psi_2^2(2\Phi_{11} - 3\eta\Psi_2)} \right) \right. \\ &\quad \left. + \frac{2\Phi_{11} - 3\eta\Psi_2}{2\eta\varphi_1} \left(2\mu \left(D \frac{\Phi_{11}}{\Psi_2} \right) - 4\rho\mu \frac{(\Phi_{11} + 2\eta\Psi_2)(\Phi_{11} + 3\eta\Psi_2)}{\Psi_2^2} \right) \right) \\ &= \frac{1}{2\eta\varphi_1} \left(-24\rho\mu \frac{\Phi_{11} - 3\eta\Psi_2}{2\Phi_{11} - 3\eta\Psi_2} \right); \end{aligned} \quad (77)$$

thus, each of the coefficients of \mathcal{O}_{2r} take the same form as the corresponding coefficients of \mathcal{O}_{1r} acting on the variable χ defined in Equation (53). As a result, we can again express the operators acting on $\Psi_3^{(1)}$ with the operators acting on $\varphi_2^{(1)}$ as a single operator acting on χ , and obtain in this way the second equation for $\Psi_4^{(1)}$ and χ ,

$$\mathbf{B}_1 \Psi_4^{(1)} + \mathbf{B}_1 \chi = \mathbf{B}_3 \kappa T_{22}^{(1)} + \frac{1}{4\varphi_1} (3\eta\Psi_2 - 2\Phi_{11}) J_2^{(1)}, \quad (78)$$

where $\mathbf{B}_1 = \mathcal{O}_{1r} + \mathcal{O}_{1a}$. We have shown that it is possible to obtain a couple of equations for $\Psi_4^{(1)}$ and χ that are independent of $\lambda^{(1)}$ and $\nu^{(1)}$. However, the complete system is not solved as long as, in order to determine the perturbed spinor coefficients, one must obtain $\Psi_4^{(1)}$, $\Psi_3^{(1)}$, and $\varphi_2^{(1)}$ independently, which is not possible within this formalism because, as mentioned above, the system is underdetermined.

4. Sources of the Perturbation and Harmonic Decomposition

Because the previous section is quite lengthy due to all the calculations, we briefly present the two important equations that describe the gravitational and electromag-

netic/gravitational perturbations for $\Psi_4^{(1)}$ and χ and further comment on the sources that may cause these perturbations. From Equations (60) and (78), we have

$$A_1 \Psi_4^{(1)} + A_5 \chi = A_4 \kappa T_{22}^{(1)}, \quad (79)$$

$$B_1 \Psi_4^{(1)} + B_2 \chi = B_3 \kappa T_{22}^{(1)} + \frac{1}{4\varphi_1} (2\Phi_{11} - 3\eta\Psi_2) J_2^{(1)}, \quad (80)$$

where

$$\chi = \eta (2\varphi_1 \Psi_3^{(1)} - 3\Psi_2 \varphi_2^{(1)}), \quad (81)$$

$$A_1 = \left(\Delta + 3\eta\mu \frac{6\Phi_{11} + 5\eta\Psi_2}{2\Phi_{11} + 3\eta\Psi_2} \right) (D_0 - \eta\rho) - 2\Phi_{11} - 3\eta\Psi_2 + \frac{2\Phi_{11} + 3\eta\Psi_2}{2\Phi_{11} - 3\eta\Psi_2} \bar{\delta}_{-2} \delta_4, \quad (82)$$

$$A_5 = -\frac{(2\varphi_1)}{(2\Phi_{11} - 3\eta\Psi_2)} \left(\Delta + 3\eta\mu \frac{6\Phi_{11} + 5\eta\Psi_2}{2\Phi_{11} + 3\eta\Psi_2} \right) \bar{\delta}_{-2}, \quad (83)$$

$$A_4 = \eta \frac{2\Phi_{11} + 3\eta\Psi_2}{2\Phi_{11} - 3\eta\Psi_2} \bar{\delta}_{-2} \bar{\delta}_0, \quad (84)$$

$$B_1 = -\frac{4\Phi_{11}}{2\Phi_{11} + 3\eta\Psi_2} \left(D_0 - \eta\rho \frac{2\Phi_{11} - 9\eta\Psi_2}{2\Phi_{11} - 3\eta\Psi_2} \right) \delta_4, \quad (85)$$

$$B_2 = \frac{1}{2\eta\varphi_1} \left(\left(D_0 - 4\eta\rho \frac{\Phi_{11} - 3\eta\Psi_2}{2\Phi_{11} - 3\eta\Psi_2} \right) (\Delta_0 + 6\eta\mu) + 2(2\Phi_{11} - 3\eta\Psi_2) + \frac{2\Phi_{11} - 3\eta\Psi_2}{2\Phi_{11} + 3\eta\Psi_2} \delta_4 \bar{\delta}_{-2} \right), \quad (86)$$

$$B_3 = -\eta \left(D_0 + 6\eta\rho \frac{\eta\Psi_2}{2\Phi_{11} - 3\eta\Psi_2} \right) \bar{\delta}_0. \quad (87)$$

As the source of perturbation, let us consider a charged dust-like matter falling radially into the black hole with stress energy tensor $T_{\mu\nu} = \rho_f u^\mu u^\nu$, where u^μ is the four velocity and ρ_f is the rest mass density of the fluid. In our analysis, we consider that the fluid is falling radially into the black hole with four velocities:

$$u^\mu = (u^t(t, r), u^r(t, r), 0, 0). \quad (88)$$

When the fluid is charged, it induces an electric current given by $J^\mu{}^{(1)} = \rho_e u_e^\mu$, where the electric density is $\rho_e = q\rho_f$, and q is the charge per unit of the mass of each particle.

An important property of the system of Equations (79) and (80) with the given form of the sources is that the system can be decoupled into an angular and radial set of equations.

First, one must notice that the different functions of the system, Equations (79) and (80), have different spin weights. For instance, the Weyl scalar $\Psi_4^{(1)}$ has a spin weight of minus two, whereas χ has spin weight of minus one. Furthermore, the rest mass density ρ_n and the density of charge ρ_e are scalar functions with zero spin. Thus, the different operators acting on these functions have to be such that they finally produce quantities with the same spin weight. Indeed, as we show below, Equation (79) has spin weight of -2 and Equation (80) has a spin weight of -1 . With this in mind and using the fact that the spin-weighted spherical harmonics $Y_s^{(l,m)}$ form a basis for each weight s , we write

$$\Psi_4^{(1)} = \sum_{l,m} \Psi_{4lm}(t, r) Y_{-2}^{(l,m)}(\theta, \varphi), \quad (89)$$

$$\chi = \sum_{l,m} \chi_{lm}(t, r) Y_{-1}^{(l,m)}(\theta, \varphi), \quad (90)$$

$$\rho_n = \sum_{l,m} \rho_{nlm}(t, r) Y_0^{(l,m)}(\theta, \varphi), \quad (91)$$

$$\rho_e = \sum_{l,m} \rho_{elm}(t, r) Y_0^{(l,m)}(\theta, \varphi). \quad (92)$$

In spherical symmetry, we can choose the vector $k^\mu = (1, -1, 0, 0)$ such that the operators $\delta_N = m^\mu \partial_\mu$ and $\bar{\delta}_N$ in Equation (30) can be written in terms of the eth and eth-bar operators, $\bar{\delta}_s = -(\partial_\theta + i \csc \theta \partial_\varphi - s \cot \theta)$ and $\bar{\delta}_s = -(\partial_\theta - i \csc \theta \partial_\varphi + s \cot \theta)$ as follows:

$$\delta_N = -\frac{1}{\sqrt{2}r} \bar{\delta}_{-\frac{N}{2}}, \quad \bar{\delta}_N = -\frac{1}{\sqrt{2}r} \bar{\delta}_{\frac{N}{2}}. \quad (93)$$

The action of the eth, eth-bar operators on $Y_s^{(l,m)}$ is to raise or lower the spin weight as follows:

$$\bar{\delta}_s Y_s^{(l,m)} = \sqrt{(l-s)(l+s+1)} Y_{s+1}^{(l,m)}, \quad (94)$$

$$\bar{\delta}_s Y_s^{(l,m)} = -\sqrt{(l+s)(l-s+1)} Y_{s-1}^{(l,m)}. \quad (95)$$

Given the action of the operators on the spin weight, one gets the following identities for the terms in the perturbation equations:

$$\begin{aligned} \bar{\delta}_{-2} \delta_4 \Psi_4^{(1)} &= \frac{1}{2r^2} \bar{\delta}_{-1} \bar{\delta}_{-2} \sum_{l,m} \Psi_{4lm} Y_{-2}^{(l,m)} = -\frac{1}{2r^2} \sum_{l,m} (l-1)(l+2) \Psi_{4lm} Y_{-2}^{(l,m)}, \\ \bar{\delta}_{-2} \chi &= -\frac{1}{\sqrt{2}r} \bar{\delta}_{-1} \sum_{l,m} \chi_{lm} Y_{-1}^{(l,m)} = -\frac{1}{\sqrt{2}r} \sum_{l,m} \sqrt{(l-1)(l+2)} \chi_{lm} Y_{-2}^{(l,m)}, \\ \bar{\delta}_{-2} \bar{\delta}_0 \kappa T_{22}^{(1)} &= \kappa \frac{(k^\mu u_\mu)^2}{2r^2} \bar{\delta}_{-1} \bar{\delta}_0 \sum_{l,m} \rho_{nlm}(t, r) Y_0^{(l,m)}, \\ &= \kappa \frac{(k^\mu u_\mu)^2}{2r^2} \sum_{l,m} \sqrt{(l-1)l(l+1)(l+2)} \rho_{nlm}(t, r) Y_{-2}^{(l,m)}, \\ \delta_4 \Psi_4^{(1)} &= -\frac{1}{\sqrt{2}r} \bar{\delta}_{-2} \sum_{l,m} \sum_{l,m} \Psi_{4lm} Y_{-2}^{(l,m)} = -\frac{1}{\sqrt{2}r} \sum_{l,m} \sqrt{(l-1)(l+2)} \Psi_{4lm} Y_{-1}^{(l,m)}, \\ \delta_4 \bar{\delta}_{-2} \chi &= \frac{1}{2r^2} \bar{\delta}_{-2} \bar{\delta}_{-1} \sum_{l,m} \chi_{lm} Y_{-1}^{(l,m)} = -\frac{1}{2r^2} \sum_{l,m} (l-1)(l+2) \chi_{lm} Y_{-1}^{(l,m)}, \\ \delta_0 \kappa T_{22}^{(1)} &= -\kappa \frac{(k^\mu u_\mu)^2}{\sqrt{2}r} \bar{\delta}_0 \sum_{l,m} \rho_{nlm} Y_0^{(l,m)} = \frac{(k^\mu u_\mu)^2}{\sqrt{2}r} \sum_{l,m} \sqrt{l(l+1)} \rho_{nlm} Y_{-1}^{(l,m)}. \end{aligned} \quad (96)$$

Thus, the angular dependence of the dynamical equations is encoded in the spin weighted spherical harmonics. Furthermore, all the terms that appear in Equation (79) have spin weights of -2 , and those appearing in Equation (80) have spin weights of -1 , which confirms that the equations are balanced with respect to the spin weight.

The angular part of the perturbation equations can be factorized using the orthogonality properties of the spherical harmonics and each equation, and can be reduced to a set of equations for each mode as follows: multiply Equation (79) by $Y_{-2}^{(l',m')}$, and Equation (79) by $Y_{-1}^{(l',m')}$; then, integrate the element of solid angle. As a result of the orthogonality of the spherical harmonics, each sum reduces and one obtains an equation for each mode (l, m) .

Finally, recalling the Peeling theorem [29] that states that the Weyl scalars have the asymptotic decay $\Psi_s \equiv 1/r^{5-s}$, it proves convenient to rewrite the equation for the gravitational perturbation in terms of the quantity $r \Psi_4^{(1)}$, which does not decay in the asymptotic region. For the electromagnetic/gravitational perturbation, the product between the Weyl and Maxwell scalars forming χ decays as $1/r^4$; thus, the product $r^4 \chi$ has a constant amplitude. For simplicity, let us define the amplitudes for each radial mode:

$$R_{lm4} = r \Psi_{lm4} \quad \text{and} \quad R_{lm3} = r^4 \chi_{lm}. \quad (97)$$

Then, the perturbation equations take the form

$$C_1 R_{lm4} + C_2 R_{lm3} = \kappa C_3 \rho_{lm} (k^\mu u_\mu)^2, \quad (98)$$

$$D_1 R_{lm4} + D_2 R_{lm3} = \kappa D_3 \rho_{lm} (k^\mu u_\mu)^2 - \frac{1}{4} (2\Phi_{11} + 3\Psi_2) \frac{\sqrt{l(l+1)}}{\sqrt{2}r} \rho_{lm} (k^\mu u_\mu), \quad (99)$$

where

$$\begin{aligned} C_1 &= \left[\left(\Delta - 3\mu \frac{6\Phi_{11} - 5\Psi_2}{2\Phi_{11} - 3\Psi_2} \right) (D + \rho - 4\epsilon) - 2\Phi_{11} + 3\Psi_2 - \frac{2\Phi_{11} - 3\Psi_2}{2\Phi_{11} + 3\Psi_2} \frac{(l-1)(l+2)}{2r^2} \right] \frac{1}{r}, \\ C_2 &= \sqrt{2(l-1)(l+2)} \frac{\varphi_1}{2\Phi_{11} + 3\Psi_2} \left(\left(\Delta - 3\mu \frac{6\Phi_{11} - 5\Psi_2}{2\Phi_{11} - 3\Psi_2} \right) \frac{1}{r} \right) \frac{1}{r^4}, \\ C_3 &= - \frac{2\Phi_{11} - 3\Psi_2}{2\Phi_{11} + 3\Psi_2} \frac{\sqrt{(l-1)l(l+1)(l+2)}}{2r^2}, \\ D_1 &= \varphi_1 \left[\sqrt{2(l-1)(l+2)} \frac{2\Phi_{11}}{2\Phi_{11} - 3\Psi_2} \left(D + \rho_s \frac{2\Phi_{11} + 9\Psi_2}{2\Phi_{11} + 3\Psi_2} - 4\epsilon \right) \frac{1}{r} \right] \frac{1}{r}, \\ D_2 &= -\frac{1}{2} \left[\left(D + 4\rho \frac{\Phi_{11} + 3\Psi_2}{2\Phi_{11} + 3\Psi_2} - 4\epsilon \right) (\Delta - 6\mu) + 2(2\Phi_{11} + 3\Psi_2) - \frac{(l-1)(l+2)}{2r^2} \frac{2\Phi_{11} + 3\Psi_2}{2\Phi_{11} - 3\Psi_2} \right] \frac{1}{r^4}, \\ D_3 &= \sqrt{\frac{l(l+1)}{2}} \varphi_1 \left(D + 6\rho \frac{\Psi_2}{2\Phi_{11} + 3\Psi_2} - 4\epsilon \right) \frac{1}{r}. \end{aligned} \quad (100)$$

Equations (98) and (99) are the final coupled dynamical equations for the gravitational and electromagnetic perturbations in the time domain, which can be written in any coordinate basis.

5. Perturbations in Reissner–Nordström Spacetime in Kerr–Schild Coordinates

Following our previous work [28], we focus on the Reissner–Nordström metric in Kerr–Schild-type coordinates:

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 + 2\left(\frac{2M}{r} - \frac{Q^2}{r^2}\right) dt dr + \left(1 + \frac{2M}{r} - \frac{Q^2}{r^2}\right) dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \quad (101)$$

where the null tetrad is

$$\begin{aligned} l^\mu &= \frac{1}{2} \left(1 + \frac{2M}{r} - \frac{Q^2}{r^2}, 1 - \frac{2M}{r} + \frac{Q^2}{r^2}, 0, 0 \right), \\ k^\mu &= (1, -1, 0, 0), \\ m^\mu &= \frac{1}{\sqrt{2}r} (0, 0, 1, i \csc\theta), \end{aligned} \quad (102)$$

and the nonvanishing components of the Weyl, Ricci, and Maxwell scalars associated to this geometry are

$$\Psi_2 = \frac{M}{r^3} - \frac{Q^2}{r^4}, \quad \Phi_{11} = \frac{Q^2}{2r^4}, \quad \varphi_1 = \frac{Q}{\sqrt{2}r^2}. \quad (103)$$

For this spacetime and with our choice of null tetrad, the nonzero spin coefficients are

$$\mu = \frac{1}{r}, \quad \rho_s = \frac{r^2 - 2Mr + Q^2}{2r^3}, \quad \epsilon = -\frac{1}{2} \left(\frac{M}{r^2} - \frac{Q^2}{r^3} \right), \quad \beta = -\alpha = -\frac{1}{2\sqrt{2}} \frac{\cot\theta}{r}. \quad (104)$$

In order to solve the perturbation equations, the first step is to write the equations in a dimensionless form. For this purpose, we recover the physical constants G , c , ϵ_0 and μ_0 . We start by defining the characteristic length of the system R_0 , then we write the radial coordinate r , as $r = R_0 \hat{r}$, with \hat{r} as a dimensionless quantity. We also define a characteristic

time, T_0 , such that $t = T_0 \hat{t}$. In terms of the dimensionless quantities, we find it useful to define the quantity

$$\sigma^2 = \frac{G M}{c^2 R_0}. \quad (105)$$

The maximum value of the charge to get an extreme black hole is $Q_{\max} = \sqrt{\frac{G}{\mu_0}} \frac{M}{c}$; therefore, we can normalize the value of the charge defining a dimensionless quantity $\hat{Q} = Q/Q_{\max}$.

The dimensionless electromagnetic and gravitational scalars are

$$\varphi_{ig} = \sqrt{\frac{G}{\mu_0}} \frac{1}{c^2} \varphi_i, \quad \varphi_1 = \frac{\sigma^2}{4 \pi R_0} \frac{\hat{Q}}{\sqrt{2} \hat{r}^2}, \quad \chi = \frac{\sigma^2}{R_0^3} \hat{\chi}, \quad (106)$$

where

$$\hat{\chi} = 3 \left(\frac{1}{\hat{r}^3} - \frac{\sigma^2 \hat{Q}^2}{\hat{r}^4} \right) \varphi_2^{(1)} - \frac{\hat{Q}^2}{2 \sqrt{2} \pi \hat{r}^2} \Psi_3^{(1)}, \quad (107)$$

and:

$$\Psi_2 = \frac{\sigma^2}{R_0^2 \hat{r}^4} (\hat{r} - \sigma^2 \hat{Q}^2), \quad \Phi_{11} = \frac{\sigma^4}{R_0^2} \frac{\hat{Q}^2}{2 \hat{r}^4}.$$

Furthermore, the dimensionless radial functions \hat{R}_4 and \hat{R}_3 are

$$\hat{R}_4 = \frac{1}{R_0} R_4, \quad \hat{R}_3 = \frac{1}{R_0} R_3. \quad (108)$$

By replacing the Weyl scalars Equation (103) and the spin coefficients Equation (104) in the perturbation Equations (98) and (99), we obtain the following system in terms of dimensionless quantities:

$$\begin{aligned} & \left\{ \left(\hat{r}^2 + 2 \sigma^2 \hat{r} - \sigma^4 \hat{Q}^2 \right) \frac{\partial^2}{\partial \hat{t}^2} - \left(\hat{r}^2 - 2 \sigma^2 \hat{r} + \sigma^4 \hat{Q}^2 \right) \frac{\partial^2}{\partial \hat{r}^2} - 2 \sigma^2 \left(2 \hat{r} - \sigma^2 \hat{Q}^2 \right) \frac{\partial^2}{\partial \hat{t} \partial \hat{r}} + \right. \\ & - 2 \frac{(6 \hat{r}^3 + (3 - 10 \hat{Q}^2) \sigma^2 \hat{r}^2 - 5 \sigma^4 \hat{Q}^2 \hat{r} - 2 \sigma^6 \hat{Q}^4)}{\hat{r} (3 \hat{r} - 4 \sigma^2 \hat{Q}^2)} \frac{\partial}{\partial \hat{t}} - 2 \frac{(6 \hat{r}^3 - (3 - 10 \hat{Q}^2) \sigma^2 \hat{r}^2 + 5 \sigma^4 \hat{Q}^2 \hat{r} + 2 \sigma^6 \hat{Q}^4)}{\hat{r} (3 \hat{r} - 4 \sigma^2 \hat{Q}^2)} \frac{\partial}{\partial \hat{r}} \\ & \left. + (l - 1) (l + 2) \frac{3 \hat{r} - 4 \sigma^2 \hat{Q}^2}{3 \hat{r} - 2 \sigma^2 \hat{Q}^2} - \frac{6 \sigma^2 (\hat{r} - 2 \sigma^2 \hat{Q}^2)}{\hat{r} (3 \hat{r} - 4 \sigma^2 \hat{Q}^2)} \right\} \hat{R}_4(\hat{t}, \hat{r}) + \\ & \frac{\sqrt{(l - 1) (l + 2)}}{2 \pi (3 \hat{r} - 2 \sigma^2 \hat{Q}^2)} \hat{Q} \left(\frac{\partial}{\partial \hat{t}} - \frac{\partial}{\partial \hat{r}} + \frac{4 \sigma^2 \hat{Q}^2}{\hat{r} (3 \hat{r} - 4 \sigma^2 \hat{Q}^2)} \right) \hat{R}_3(\hat{t}, \hat{r}) = \\ & 3 \sigma^2 \hat{r} \sqrt{(l - 1) l (l + 1) (l + 2)} \frac{3 \hat{r} - 4 \sigma^2 \hat{Q}^2}{3 \hat{r} - 2 \sigma^2 \hat{Q}^2} \hat{\rho}(\hat{t}, \hat{r}) (\hat{u}_\mu k^\mu)^2, \end{aligned} \quad (109)$$

and

$$\begin{aligned}
& \frac{\sigma^4 \hat{Q}^3 \sqrt{(l-1)(l+2)}}{2\pi(3\hat{r}-4\sigma^2\hat{Q}^2)} \left[\left(\hat{r}^2 + 2\sigma^2\hat{r} - \sigma^4\hat{Q}^2 \right) \frac{\partial}{\partial \hat{t}} + \left(\hat{r}^2 - 2\sigma^2\hat{r} + \sigma^4\hat{Q}^2 \right) \frac{\partial}{\partial \hat{r}} + \right. \\
& \left. \frac{3\hat{r}^2(\hat{r}+2\sigma^2) - \sigma^2\hat{Q}^2(4\hat{r}^2+9\sigma^2\hat{r}-\sigma^4\hat{Q}^2)}{\hat{r}(3\hat{r}-2\sigma^2\hat{Q}^2)} \right] \hat{R}_4(\hat{t}, \hat{r}) \\
& + \left\{ \left(\hat{r}^2 + 2\sigma^2\hat{r} - \sigma^4\hat{Q}^2 \right) \frac{\partial^2}{\partial \hat{t}^2} - \left(\hat{r}^2 - 2\sigma^2\hat{r} + \sigma^4\hat{Q}^2 \right) \frac{\partial^2}{\partial \hat{r}^2} - 2\sigma^2(2\hat{r} - \sigma^2\hat{Q}^2) \frac{\partial^2}{\partial \hat{t} \partial \hat{r}} \right. \\
& - 2 \frac{3\hat{r}^3 - \sigma^2\hat{Q}^2\hat{r}^2 + \sigma^4\hat{Q}^2\hat{r} - \sigma^6\hat{Q}^4}{\hat{r}(3\hat{r}-2\sigma^2\hat{Q}^2)} \frac{\partial}{\partial \hat{t}} - 2 \frac{(\hat{r} - \sigma^2\hat{Q}^2)(3\hat{r}^2 - \sigma^4\hat{Q}^2)}{\hat{r}(3\hat{r}-2\sigma^2\hat{Q}^2)} \frac{\partial}{\partial \hat{r}} \\
& \left. + (l-1)(l+2) \frac{3\hat{r}-2\sigma^2\hat{Q}^2}{3\hat{r}-4\sigma^2\hat{Q}^2} + 2 \frac{3\hat{r}^2 - \sigma^4\hat{Q}^2}{\hat{r}(3\hat{r}-2\sigma^2\hat{Q}^2)} \right\} \hat{R}_3(\hat{t}, \hat{r}) = \\
& - \frac{3\sigma^4\hat{r}\sqrt{l(l+1)}\hat{Q}}{2\pi} \left[\left(r^2 + 2Mr - Q^2 \right) \frac{\partial}{\partial \hat{t}} + \left(r^2 - 2Mr + Q^2 \right) \frac{\partial}{\partial \hat{r}} \right. \\
& \left. + \frac{2\hat{r}^3 + \sigma^2(8 - \hat{Q}^2)\hat{r}^2 - 16\sigma^4\hat{Q}^2\hat{r} + 7\sigma^6\hat{Q}^4}{\hat{r}(3\hat{r}-2\sigma^2\hat{Q}^2)} \right] \hat{\rho}(\hat{t}, \hat{r}) (\hat{u}_\mu k^\mu)^2 - \frac{3\hat{r}\sqrt{2l(l+1)}}{4\pi} (3\hat{r} - 2\sigma^2\hat{Q}^2) \hat{\rho}_e \hat{u}_{e\mu} k^\mu,
\end{aligned} \tag{110}$$

where, for simplicity in the notation, we dropped the indices l, m on each mode.

5.1. Matter Models

We shall consider that there are two sources of matter that cause a perturbation in the black hole: one associated with neutral matter characterized by the rest mass density $\hat{\rho}_n$ and the other associated to charged particles with density $\hat{\rho}_e$. We consider that neutral particles only move in the radial direction and are free falling into the black hole. The conservation of the number of particles $\nabla_\mu J^\mu = 0$, where $J^\mu = \rho_n u_n^\mu$, for the metric Equation (101) gives

$$\frac{\partial \hat{\rho}_n}{\partial \hat{t}} + v_n^r \frac{\partial \hat{\rho}_n}{\partial \hat{r}} + \hat{\rho}_n \frac{4(\hat{u}_n^r)^2 + \hat{r} \frac{\partial (\hat{u}_n^r)^2}{\partial \hat{r}}}{2\hat{r}(\hat{u}_n^r)^2} = 0, \tag{111}$$

where we defined the three velocity as $v_n^r = \frac{\hat{u}_n^r}{\hat{u}_n^0}$ and we assumed that the dimensionless four velocity has components $\hat{u}_n^\mu = (\hat{u}_n^0, \hat{u}_n^r, 0, 0)$. For charged particles, we make the same assumptions, so we obtain that the conservation of the number of particles implies

$$\frac{\partial \hat{\rho}_e}{\partial \hat{t}} + v_e^r \frac{\partial \hat{\rho}_e}{\partial \hat{r}} + \hat{\rho}_e \frac{4(\hat{u}_e^r)^2 + \hat{r} \frac{\partial (\hat{u}_e^r)^2}{\partial \hat{r}}}{2\hat{r}(\hat{u}_e^r)^2} = 0. \tag{112}$$

Considering the normalization of the four velocity $\hat{u}^\mu \hat{u}_\mu = -1$ for both types of particles, we can express \hat{u}^0 in terms of \hat{u}^r as [28]

$$\hat{u}^0 = \frac{\sigma^2 \left(2 - \frac{\sigma^2 \hat{Q}^2}{\hat{r}} \right) \frac{\hat{u}^r}{\hat{r}} + \frac{1}{\sigma} \sqrt{\sigma^2 \hat{u}^2 - k \hat{\Delta}}}{\hat{\Delta}}, \tag{113}$$

where $\hat{\Delta} = 1 - \frac{2\sigma^2}{\hat{r}} + \frac{\sigma^4 \hat{Q}^2}{\hat{r}^2}$. As previously stated, this expression is valid for neutral and charged particles.

For spherically symmetric static spacetimes, one can determine the motion of test particles up to quadrature by means of the Euler–Lagrange equations in the following way: Let us consider the dimensionless Lagrangian for neutral particles

$$\hat{\mathcal{L}} = \frac{1}{2} (g_{\mu\nu} \hat{u}^\mu \hat{u}^\nu), \tag{114}$$

using the Euler–Lagrange equation for x^0 and the metric Equation (101), one obtains a conserved quantity associated to the energy of the particles at infinity $\hat{\epsilon}_n$ and, consequently,

$$\hat{u}_n^0 = \frac{\hat{\epsilon}_n + \sigma^2 \left(2 - \frac{\sigma^2 \hat{Q}^2}{\hat{r}}\right) \frac{\hat{u}_n^r}{\hat{r}}}{\hat{\Delta}}, \quad (115)$$

where $\hat{\epsilon}_n = -\frac{\partial \hat{\mathcal{L}}}{\partial \hat{u}_n^0}$. Proceeding in a similar way for charged particles, the dimensionless Lagrangian is

$$\hat{\mathcal{L}}_e = \frac{1}{2} \left(\hat{u}^\mu \hat{u}_\mu + 2 \sigma^2 \hat{q} \hat{Q} \frac{M}{m} \frac{\hat{u}^0}{\hat{r}} \right), \quad (116)$$

where the vector potential was taken as $A_\mu = \left(\frac{Q}{\epsilon_0 r}, \vec{0} \right) = \left(\frac{Q_{\max} \mu_0 c^2}{R_0} \frac{\hat{Q}}{\hat{r}}, \vec{0} \right)$ and we have written $q = Q_{\max} \hat{q}$. From the staticity of the spacetime via the Euler–Lagrange equation for t , we get

$$\hat{u}_e^0 = \frac{\hat{\epsilon}_e + \sigma^2 \left(2 - \frac{\sigma^2 \hat{Q}^2}{\hat{r}}\right) \frac{\hat{u}_e^r}{\hat{r}} - \frac{\lambda}{2\hat{r}}}{\hat{\Delta}}, \quad (117)$$

where $\hat{\epsilon}_e = -\frac{\partial \hat{\mathcal{L}}_e}{\partial \hat{u}_e^0}$ and $\lambda = 2 \sigma^2 \hat{q} \hat{Q} \frac{M}{m}$. Using Equation (113) in both Equations (115) and (117), we obtain

$$(\hat{u}_e^r)^2 = \left(\hat{\epsilon}_e - \frac{\lambda}{2\hat{r}} \right)^2 - \frac{\hat{\Delta}}{\sigma^2}, \quad (118)$$

$$(\hat{u}_n^r)^2 = \hat{\epsilon}_n^2 - \frac{\hat{\Delta}}{\sigma^2}. \quad (119)$$

Finally, the projection of the four velocity along the null vector $k^\mu u_\mu$ for both types of particles provides a couple of expressions that are useful: $\hat{u}_e^0 + \hat{u}_e^r = \frac{\hat{\epsilon}_e - \frac{\lambda}{2\hat{r}} + \hat{u}_e^r}{\hat{\Delta}}$, and $\hat{u}_n^0 + \hat{u}_n^r = \frac{\hat{\epsilon}_n + \hat{u}_n^r}{\hat{\Delta}}$.

Given the components of the velocity for both neutral and charged particles and the evolution of the densities, one can numerically solve the perturbation equations given an initial distribution of matter and compute the resultant gravitational signal due to the gravitational and electromagnetic perturbation induced in the black hole.

5.2. First-Order Reduction and Numerical Implementation

In terms of the 3+1 decomposition of the spacetime $ds^2 = -(\alpha^2 - \beta^r \beta_r) dt^2 + 2 \beta_r dr dt + \gamma_{rr}^2 dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2)$, one can determine the lapse function α , the shift vector β^r , and the metric component γ_{rr} from metric Equation (101), yielding

$$\alpha = \left(1 + \frac{2 \sigma^2}{\hat{r}} - \frac{\sigma^4 \hat{Q}^2}{\hat{r}^2} \right)^{-1/2}, \quad \beta_r = \frac{2 \sigma^2}{\hat{r}} - \frac{\sigma^4 \hat{Q}^2}{\hat{r}^2}, \quad \beta^r = \alpha^2 \beta_r, \quad \gamma_{rr} = \left(1 - \frac{2 \sigma^2}{\hat{r}} + \frac{\sigma^4 \hat{Q}^2}{\hat{r}^2} \right)^{1/2}. \quad (120)$$

In order to write the system of second-order differential Equations (98) and (99) as a first-order system suitable for numerical integration, we introduce the auxiliary functions

$$\hat{\pi}_a \equiv \frac{1}{\alpha^2} (\partial_{\hat{t}} \hat{R}_a - \beta^r \hat{\psi}_a), \quad \psi_a \equiv \partial_{\hat{r}} \hat{R}_a, \quad \text{with } a = 3, 4. \quad (121)$$

From the definition of $\hat{\pi}_a$, one obtains the time evolution of \hat{R}_a ; however, by taking the time derivative of $\hat{\psi}_a$ and replacing the lapse and shift, one gets

$$\partial_{\hat{t}} \hat{R}_a = \frac{\hat{r}^2 \hat{\pi}_a + \sigma^2 (2\hat{r} - \sigma^2 \hat{Q}^2) \hat{\psi}_a}{\hat{r}^2 + 2\sigma^2 \hat{r} - \sigma^4 \hat{Q}^2}, \quad (122)$$

$$\partial_{\hat{t}} \hat{\psi}_a = \frac{\hat{r}^2 \partial_{\hat{r}} \hat{\pi}_a + \sigma^2 (2\hat{r} - \sigma^2 \hat{Q}^2) \partial_{\hat{r}} \hat{\psi}_a}{\hat{r}^2 + 2\sigma^2 \hat{r} - \sigma^4 \hat{Q}^2} + \frac{2\hat{r} \sigma^2 (\hat{r} - \sigma^2 \hat{Q}^2)}{(\hat{r}^2 + 2\sigma^2 \hat{r} - \sigma^4 \hat{Q}^2)^2} (\hat{\pi}_a - \hat{\psi}_a). \quad (123)$$

The equations for $\hat{\pi}_4$ and $\hat{\pi}_3$ are obtained by substituting first-order functions into Equations (109) and (110):

$$\begin{aligned} \partial_{\hat{t}} \hat{\pi}_4 = & \frac{\sigma^2 (2\hat{r} - \sigma^2 \hat{Q}^2) \partial_{\hat{r}} \hat{\pi}_4 + \hat{r}^2 \partial_{\hat{r}} \hat{\psi}_4}{(\hat{r}^2 + 2\sigma^2 \hat{r} - \sigma^4 \hat{Q}^2)} \\ & + 2 \frac{3\hat{r}^3 (2\hat{r}^2 + 5\sigma^2 \hat{r} + 4\sigma^4) - \sigma^2 \hat{Q}^2 (10\hat{r}^4 + 31\sigma^2 \hat{r}^3 + 2\sigma^4 (15 - 4\hat{Q}^2) \hat{r}^2 - 2\sigma^6 \hat{Q}^2 (8\hat{r} - \sigma^2 \hat{Q}^2))}{\hat{r} (\hat{r}^2 + 2\sigma^2 \hat{r} - \sigma^4 \hat{Q}^2)^2 (3\hat{r} - 4\sigma^2 \hat{Q}^2)} \hat{\pi}_4 \\ & + 2 \frac{3\hat{r}^3 (2\hat{r}^2 + 11\sigma^2 \hat{r} + 12\sigma^4) + \sigma^2 \hat{Q}^2 (10\hat{r}^4 + 7\sigma^2 \hat{r}^3 - 2\sigma^4 (9 + 4\hat{Q}^2) \hat{r}^2 - 2\sigma^6 \hat{Q}^2 (2\hat{r} - \sigma^2 \hat{Q}^2))}{\hat{r} (\hat{r}^2 + 2\sigma^2 \hat{r} - \sigma^4 \hat{Q}^2)^2 (3\hat{r} - 4\sigma^2 \hat{Q}^2)} \hat{\psi}_4 \\ & - \left(\frac{(l-1)(l+2) (3\hat{r} - 4\sigma^2 \hat{Q}^2)}{\hat{r}^2 (3\hat{r} - 2\sigma^2 \hat{Q}^2)} - \frac{6\sigma^2 (\hat{r} - 2\sigma^2 \hat{Q}^2)}{\hat{r}^3 (3\hat{r} - 4\sigma^2 \hat{Q}^2)} \right) \hat{R}_4 \\ & - \frac{\hat{Q} \sqrt{(l-1)(l+2)}}{2\pi (3\hat{r} - 2\sigma^2 \hat{Q}^2)} \left(\frac{\hat{\pi}_3 - \hat{\psi}_3}{(\hat{r}^2 + 2\sigma^2 \hat{r} - \sigma^4 \hat{Q}^2)} + 4 \frac{\sigma^2 \hat{Q}^2}{\hat{r}^3 (3\hat{r} - 4\sigma^2 \hat{Q}^2)} \hat{R}_3 \right) \\ & + 3\sigma^2 \sqrt{(l-1)l(l+1)(l+2)} \frac{3\hat{r} - 4\sigma^2 \hat{Q}^2}{\hat{r} (3\hat{r} - 2\sigma^2 \hat{Q}^2)} \rho_n(\hat{t}, \hat{r}) (\hat{u}_n^0 + \hat{u}_n^r)^2. \end{aligned} \quad (124)$$

$$\begin{aligned} \partial_{\hat{t}} \hat{\pi}_3 = & \frac{\sigma^2 (2\hat{r} - \sigma^2 \hat{Q}^2) \partial_{\hat{r}} \hat{\pi}_3 + \hat{r}^2 \partial_{\hat{r}} \hat{\psi}_3}{\hat{r}^2 + 2\sigma^2 \hat{r} - \sigma^4 \hat{Q}^2} \\ & + 2 \frac{3\hat{r}^3 (\hat{r}^2 + 2\sigma^2 \hat{r} + 2\sigma^4) - \sigma^2 \hat{Q}^2 (\hat{r}^4 + 4\sigma^2 \hat{r}^3 + 11\sigma^4 \hat{r}^2 - \sigma^6 \hat{Q}^2 (6\hat{r} - \sigma^2 \hat{Q}^2))}{\hat{r} (\hat{r}^2 + 2\sigma^2 \hat{r} - \sigma^4 \hat{Q}^2)^2 (3\hat{r} - 2\sigma^2 \hat{Q}^2)} \hat{\pi}_3 \\ & + 2 \frac{3\hat{r}^3 (\hat{r}^2 + 6\sigma^2 \hat{r} + 6\sigma^4) - \sigma^2 \hat{Q}^2 (3\hat{r}^4 + 24\sigma^2 \hat{r}^3 + (29 - 8\hat{Q}^2) \sigma^4 \hat{r}^2 - \sigma^6 \hat{Q}^2 (16\hat{r} - 3\sigma^2 \hat{Q}^2))}{\hat{r} (\hat{r}^2 + 2\sigma^2 \hat{r} - \sigma^4 \hat{Q}^2)^2 (3\hat{r} - 2\sigma^2 \hat{Q}^2)} \hat{\psi}_3 \\ & - \left(\frac{(l-1)(l+2) (3\hat{r} - 2\sigma^2 \hat{Q}^2)}{\hat{r}^2 (3\hat{r} - 4\sigma^2 \hat{Q}^2)} + \frac{2 (3\hat{r}^2 - \sigma^4 \hat{Q}^2)}{\hat{r}^3 (3\hat{r} - 2\sigma^2 \hat{Q}^2)} \right) \hat{R}_3 \\ & - \frac{\sigma^4 \hat{Q}^3 \sqrt{(l-1)(l+2)}}{2\pi (3\hat{r} - 4\sigma^2 \hat{Q}^2)} \left(\hat{\pi}_4 + \hat{\psi}_4 + \frac{3\hat{r} (\hat{r}^2 + 2\sigma^2 \hat{r} - 3\sigma^4 \hat{Q}^2) - \sigma^2 \hat{Q}^2 (4\hat{r}^2 - \sigma^4 \hat{Q}^2)}{\hat{r}^3 (3\hat{r} - 2\sigma^2 \hat{Q}^2)} \hat{R}_4 \right) \\ & + \frac{3\sigma^4 \hat{Q} \sqrt{l(l+1)}}{2\pi \hat{r}} (\hat{u}^t + \hat{u}^r)^2 \left[\left((\hat{r}^2 + 2\sigma^2 \hat{r} - \sigma^4 \hat{Q}^2) \partial_{\hat{t}} \hat{\rho} + (\hat{r}^2 - 2\sigma^2 \hat{r} + \sigma^4 \hat{Q}^2) \partial_{\hat{r}} \hat{\rho} \right) \right. \\ & \left. + \hat{\rho} \left(2 (\hat{r}^2 - 2\sigma^2 \hat{r} + \sigma^4 \hat{Q}^2) \frac{\partial_{\hat{r}} (\hat{u}^t + \hat{u}^r)}{(\hat{u}^t + \hat{u}^r)} + \frac{2\hat{r}^3 + \sigma^2 (8 - \hat{Q}^2) \hat{r}^2 - 16\sigma^4 \hat{Q}^2 \hat{r} + 7\sigma^6 \hat{Q}^4}{\hat{r} (3\hat{r} - 2\sigma^2 \hat{Q}^2)} \right) \right] + \\ & \frac{3\sigma^4 \hat{Q} \sqrt{2l(l+1)} (3\hat{r} - 2\sigma^2 \hat{Q}^2)}{4\pi \hat{r}} \hat{\rho}_e (\hat{u}_e^0 + \hat{u}_e^r). \end{aligned} \quad (125)$$

5.3. Waveforms

For our simulations, we solved the system of equations for the gravitational perturbation and the electromagnetic/gravitational perturbation with sources using Equations (124) and (125). For the source, we are considering nonspherical shell of particles falling into the hole. The numerical code evolves the first-order variables with a third-order Runge-Kutta integrator with a fourth-order spatial stencil. For a more detailed description of the code see [28]. As initial data, we used a Gaussian packet in the density describing a nonspherical shell of particles falling into the hole $\rho_n(0, \hat{r}) = \rho_0 e^{(\hat{r} - \hat{r}_{cg})^2 / 2\sigma_g^2} = \rho_e(0, \hat{r})$; where $\rho_0 = 5 \times 10^{-3}$, $\hat{r}_{cg} = 50$ and $\sigma_g = 0.8$. For the simulation, we used Kerr-Schild-type coordinates, r_{min} lies inside the event horizon, and we chose $\hat{r}_{min} = 1.5$ and $\hat{r}_{max} = 1000$. The gravitational and electromagnetic/gravitational waveforms produced by the infalling matter were extracted at a fixed $\hat{r} = 100$ radius. The gravitational and electromagnetic/gravitational functions, \hat{R}_4 , \hat{R}_3 , were initially set to zero, as were their time deriva-

tives. The outer boundary was set far enough from the horizon to ensure that any possible incoming radiation had no effect on our results.

In Figure 1a,b, we show the radial profile for the gravitational and the electromagnetic/gravitational component for the mode $l = 2$, respectively. The simulation was performed for different values of charge $\hat{Q} = 0.0, 0.5, 0.9, 0.95$, and 0.97 . We can observe that the waveforms are quite similar in structure and that the amplitude changes for different values of \hat{Q} . The gravitational waveform amplitude is greater and presents more oscillations than that of the electromagnetic/gravitational case. It is interesting to observe that the amplitude decreases for greater values of \hat{Q} in the gravitational case. On the other hand, in the electromagnetic/gravitational case, the amplitude increases for greater values of \hat{Q} . The time response for both gravitational and electromagnetic/gravitational waveforms starts and finishes at almost the same times.

In Figure 2, we plotted the absolute value on a logarithmic scale to show the different stages of the signal: the initial burst due to the initial data, the quasinormal ringing, and the tail. Figure 2a shows the gravitational signal and Figure 2b shows the electromagnetic/gravitational signal. We found no sign of mixing between gravitational and electromagnetic/gravitational frequencies. Each signal displays its characteristic ring-down frequency and the power-law decay. The ring-down frequencies were extracted from $\hat{t} \sim 50$ to $\hat{t} \sim 350$ for the gravitational case and from $\hat{t} \sim 50$ to 400 for the electromagnetic/gravitational case. The frequency of the gravitational waves is the one associated with the quadrupolar quasinormal mode. In order to find the frequencies, we fitted the data with a sinusoidal waveform. The numerical values of the corresponding frequencies are shown in Table 1. The values we found are consistent with the values given in [21]. Although it is known that quasinormal mode frequencies are complex, we were interested in the oscillatory behavior of the signal. For black holes with masses of $10M_{\odot} < M < 10^3 M_{\odot}$, the electromagnetic frequencies are in the interval 8–800 Hz, whereas the gravitational waves produced for such a range of masses are in the interval of 12–1.2 kHz [28]. As has been pointed out, quasinormal ringing can be used to determine the intrinsic properties of the black hole [30]. Electromagnetic waves with such low frequencies, however, could be easily absorbed by the interstellar medium during propagation, making it almost impossible to detect them directly.

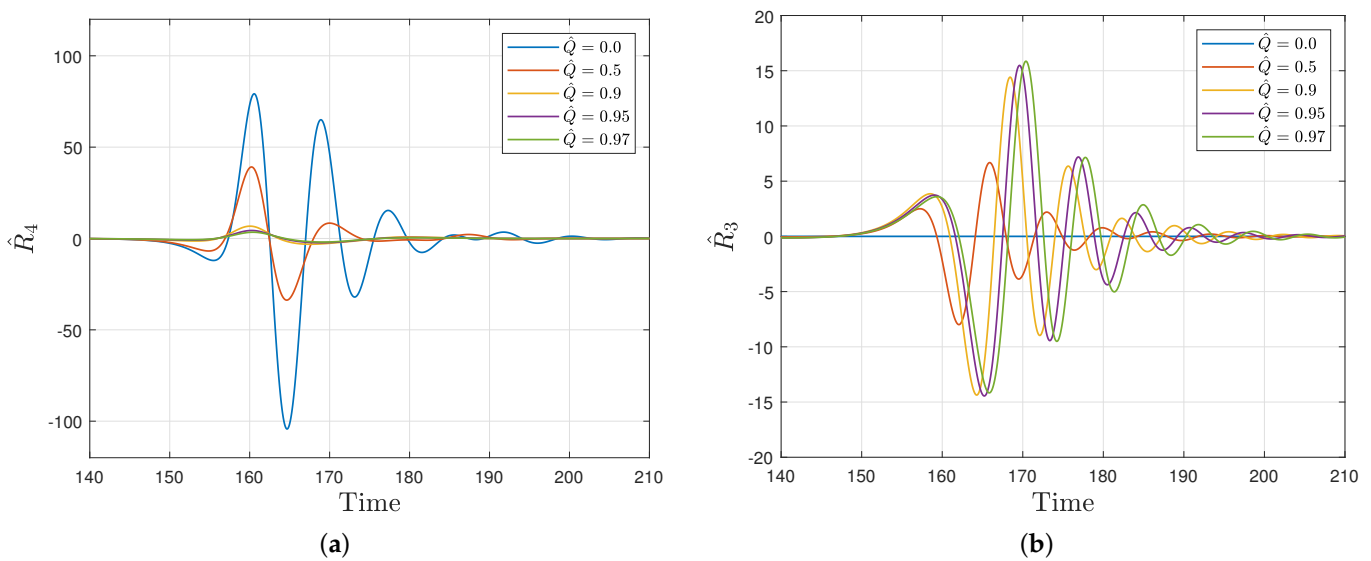


Figure 1. Radial profiles of the gravitational signal \hat{R}_4 (a), and electromagnetic/gravitational \hat{R}_3 (b) signals for the quadrupolar $l = 2$ mode with different values of $\hat{Q} = 0.0, 0.5, 0.9, 0.95$, and 0.97 .

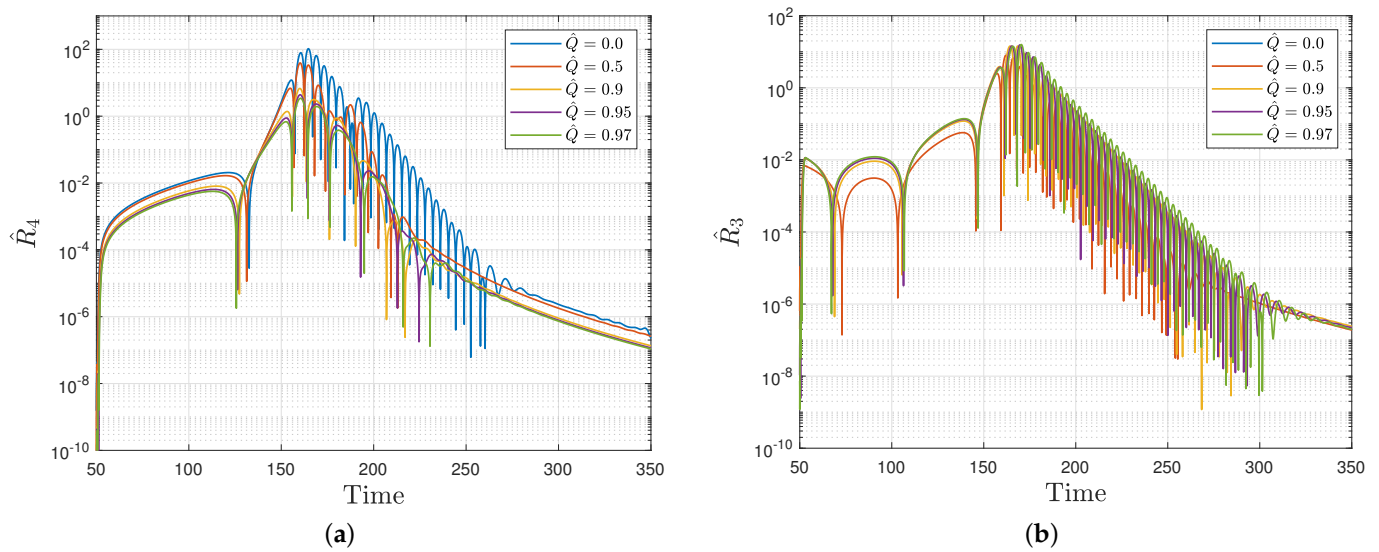


Figure 2. The gravitational \hat{R}_4 and electromagnetic/gravitational \hat{R}_3 signals in logarithmic scale for different values of charge $\hat{Q} = 0.0, 0.5, 0.9, 0.95$, and 0.97 . The signals show the characteristic phases: the initial burst, the quasinormal ringing, the power-law decay, and the tail.

Table 1. Frequency of the quasinormal modes of the black hole produced by the perturbation of the accreting matter. The frequencies are consistent with the values given in [21].

\hat{Q}	ℓ	$\hat{\omega}$
0	2	0.3736
0.2	2	0.3747
0.9	2	0.4135
0.95	2	0.4216

Next, we present the behavior of the energy carried by the gravitational wave for several values of the charge of the infalling matter for the $l = 2$ mode (any $-l \leq m \leq l$), obtained from the energy loss formula, i.e., the power of the gravitational wave, $P_{\text{gw}} = \frac{dE_{\text{gw}}}{dt}$, (see [27]):

$$P_{\text{gw}} = \lim_{r \rightarrow \infty} \frac{1}{16\pi} \int_{-\infty}^t |R_4|^2 dt'. \quad (126)$$

From Figure 3, we notice how the flux of energy carried by the gravitational wave $\Psi_4^{(1)}$ reaches a constant value. This value decreases for large values of the charge of the black hole. Although the same integral can be made for the gauge invariant quantity χ , as long as it is coupled to the gravitational radiation through $\Psi_3^{(1)}$, its classification as electromagnetic energy cannot be immediately made; thus, it deserves a deeper discussion. Here, we only mention that it is zero for $\hat{Q} = 0$ and opposite in behavior to dE_{gw}/dt ; and its asymptotic value increases with \hat{Q} .

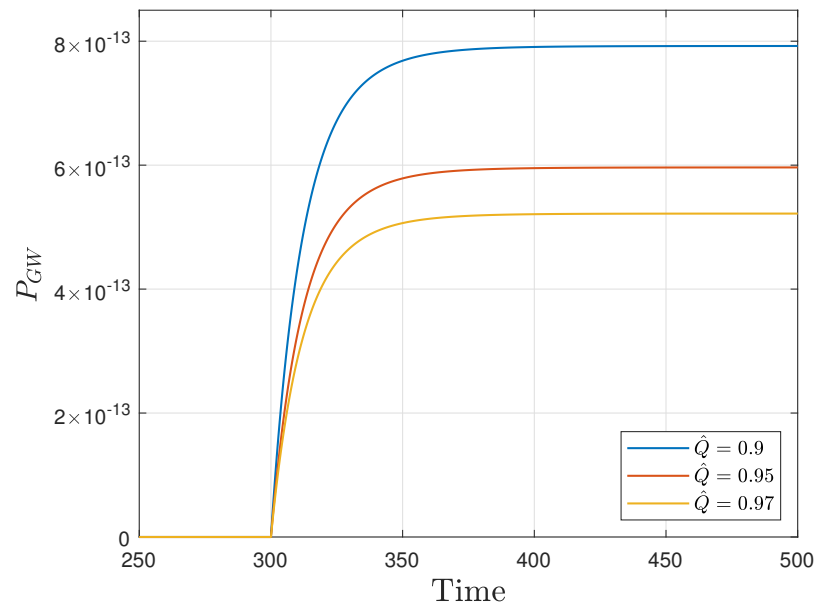


Figure 3. Energy carried by the gravitational wave, $\Psi_4^{(1)}$, according to Equation (126), for the values $\hat{Q} = 0.9, 0.95, 0.97$ proportional to the charge of the black hole.

Our system of equations allows for a purely electromagnetic/gravitational response (encoded in the function χ). Indeed, for the $l = 1$ case, which corresponds to a dipole angular distribution, there is no gravitational response, as the gravitational waves occur starting from the quadrupolar angular distribution [31], but there might be an electromagnetic one. We see that this is the case in the following: in solving the system of equations with $l = 1$, we only obtain a response in χ , which we present in Figure 4.

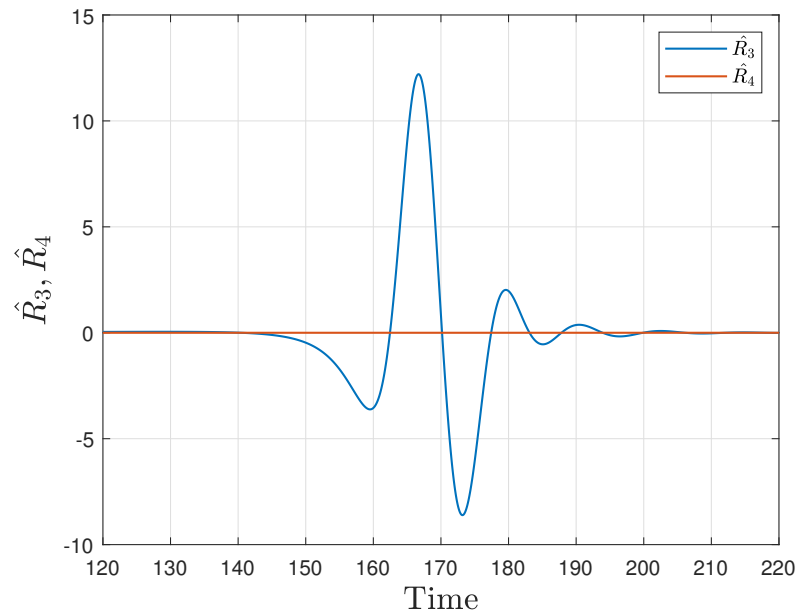


Figure 4. Waveforms for dipole perturbation, $l = 1$, with $\hat{Q} = 0.9$, $M = 1$, and $\hat{q} = -0.8$. Data were collected at $r_{\text{obs}} = 100$. Notice how in this case, there is no gravitational response, $\Psi_4^{(1)} = 0$, as there should be, but there is the response associated with χ .

6. Final Remarks

In this work, we revisited the gravitational and electromagnetic perturbations in a Reissner–Nordström black hole by means of the Newman–Penrose formalism. In our

analysis, we include the sources that cause the perturbations and discuss the particular case of a charged perfect fluid falling radially into the black hole. Using both Maxwell equations for the Maxwell scalars and the Bianchi identities for the Weyl scalars, we found a system of coupled equations for the gravitational and electromagnetic perturbations without choosing a specific gauge. A common practice to study electromagnetic and gravitational perturbations within the Newman–Penrose formalism is to choose the so-called phantom gauge (imposing $\varphi_2^{(1)} = 0$), since using this gauge, one can obtain a subsystem of equations for the perturbation of the Weyl scalars $\Psi_4^{(1)}$ and $\Psi_3^{(1)}$. However, although convenient, this choice is not unique. In this work, we show that a similar system of equations can be obtained for $\Psi_4^{(1)}$ and $\chi = 2\varphi_1\Psi_3^{(1)} - 3\Psi_2\varphi_2^{(1)}$, which involves perturbations of the electromagnetic field $\varphi_2^{(1)}$ and perturbations of the electromagnetic part of the Weyl tensor $\Psi_3^{(1)}$. Our results thus open up the possibility to explore electromagnetic and gravitational perturbations without any a priori assumption regarding the value of any of the scalars.

It is remarkable that it is not possible to obtain a perturbation equation for $\Psi_3^{(1)}$ independently of $\varphi_2^{(1)}$ without choosing a particular gauge, and only the combination given by χ can be determined via this formalism. However, this fact is not related to any physical property of the fields since one can always obtain the fields by numerically solving the Einstein–Maxwell field equations and computing all the gravitational scalars at each time step. The actual physical meaning of such a constraint in the Newman–Penrose formalism is an ongoing focus of research.

We also considered a dust-like charged fluid as a source of the perturbations. We used the spin-weighted spherical harmonics as a basis to expand the functions and obtain a system of partial differential equations for the temporal and radial components, leaving all the angular dependence of the functions on the respective basis of spherical harmonics. The resulting system of partial differential equations constitutes a hyperbolic system that can be solved numerically by standard means. In this way, we see that we have a robust procedure which sets the basis to accurately determine the simultaneous generation of gravitational and electromagnetic waveforms. A thorough study comparing amplitudes, frequencies, harmonic dependence, and power between the gravitational and the electromagnetic/gravitational signals for several fiducial values of the parameters will allow for the determination of correlations between these waveforms. This will not only give a better understanding of the process, but might also shed light on the multimessenger program in a general sense, as long as it is possible to extrapolate the correlations found in the system presented in this work to other scenarios in which signals of different interactions are generated.

Furthermore, our analysis can be used as a simple model to describe the correlation that exists between electromagnetic and gravitational wave signals that occur during the accretion of charged matter around a compact object. This is because the frequency of the gravitational waves resulting from the quasinormal ringing of black holes of $(10\text{--}10^3)M_\odot$ with moderate charge lies within the range of sensitivity of current ground-based gravitational wave interferometers.

Finally, we would like to remark that studies performed in a Reissner–Nordström spacetime frequently give valuable insights into Kerr geometry. The relationships arising from the interaction of the electromagnetic field of matter with the charge of a black hole might have a similarity with the interaction of the angular momentum of the accreting matter with the angular momentum of the black hole. The results and derivations presented in this work might prove to be useful in the perturbation analysis generated by accreting rotating matter in a Kerr background. Further studies are currently underway.

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