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Bateman Oscillators: Caldirola-Kanai and Null Lagrangians and Gauge Functions

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Abstract: The Lagrange formalism is developed for Bateman oscillators, which includes both damped and amplified systems, and a novel method to derive the Caldirola-Kanai and null Lagrangians is presented. For the null Lagrangians, the corresponding gauge functions are obtained. It is shown that the gauge functions can be used to convert the undriven Bateman oscillators into the driven ones. Applications of the obtained results to quantizatation of the Bateman oscillators are briefly discussed.

Keywords: Bateman oscillators; Lagrangian formalism; null Lagrangians; gauge functions; forces

1. Introduction

The Bateman model consists of two uncoupled oscillators, damped or time-forward and amplified or time-reversed [1], which are called here the Bateman oscillators. The equations of motion for these oscillators are derived from one Lagrangian that is known as the Bateman Lagrangian [1,2] and used in studies of the damped harmonic oscillator and its quantization [3,4]. General solutions for the equations of motion for the Bateman model are well-known and given in terms of elementary functions [5].

We solve the inverse problem of the calculus of variations [6] for the Bateman oscillators and develop a novel method to derive the standard and null Lagrangians. The standard Lagrangians (SLs) contain the square of the first order derivative of the dependent variable (the kinetic energy-term) and the square of dependent variable (the potential energy-like term), and among the derived SLs, the Caldirola-Kanai (CK) Lagrangian is obtained [7,8] and its validity to describe the Bateman oscillators is discussed. The null (or trivial) Lagrangians (NLs), when substituted into the Euler-Lagrange (EL) equation, make this equation vanish identically, which means that no equation of motion is obtained from the NLs. The NLs are also required to be expressed as the total derivative of a scalar function [9–13], which is called a gauge function [14,15].

Our objective is to derive the SLs, NLs and their gauge functions (GFs) for the Bateman oscillators. Since these oscillators are non-conservative systems, the derived Lagrangians are not consistent with the Helmholtz conditions [2,16], which guarantee the existence of Lagrangians for the conservative systems. We present a different view on these conditions and their validity for damped (amplified) oscillators. One obtained SL is the CK Lagrangian, which is well-known, and all other NLs and GFs that are derived simultaneously with the CK Lagrangian are new. The CK Lagrangian is modified by taking into account the GFs and it is shown that this modification allows for the conversion of the undriven Bateman oscillators into driven ones, which is our main result.

The outline of the paper is as follows: the equations of motion for the Bateman oscillators, their standard and null Lagrangians, and the gauge functions are derived in Section 2; derivation of the modified Caldirola-Kanai Lagrangian and the conversion of undriven oscillatory systems and driven ones are presented and discussed in Section 3; our conclusions are given in Section 4.

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2. Lagrangians and Gauge Functions for Bateman Oscillators

2.1. Equations of Motion

The original paper by Bateman [1] considers a non-conservative system for which the following Lagrangian is proposed

$$L_B[\dot{x}(t), \dot{y}(t), x(t), y(t)] = m\dot{x}(t)\dot{y}(t) + \frac{\gamma}{2}[x(t)\dot{y}(t) - \dot{x}(t)y(t)] - kx(t)y(t) , \tag{1}$$

where $x(t)$ and $y(t)$ are coordinate variables and $\dot{x}(t)$ and $\dot{y}(t)$ are their derivatives with respect to time t. In addition, m is mass and γ and k are damping and spring constants, respectively. This Lagrangian is now known as the Bateman Lagrangian [2–4] and its substitution into the EL equation for $y(t)$ and $x(t)$ gives respectively the resulting equation of motion for the damped oscillator

$$m\ddot{x}(t) + \gamma\dot{x}(t) + kx(t) = 0 , \tag{2}$$

and the equation of motion for the amplified oscillator

$$m\ddot{y}(t) - \gamma\dot{y}(t) + ky(t) = 0 . \tag{3}$$

The equations are uncoupled but they are related to each other by the transformation $[x(t), y(t), \gamma] \to [y(t), x(t), -\gamma]$, which allows replacing Equation (2) by Equation (3) and vice versa.

The equations of motion for the damped and amplified oscillators are also considered in this paper but as two unrelated dynamical systems for which the SLs, NLs and GFs are independently derived; once the SLs are obtained for both oscillators, they will be compared to the Bateman Lagrangian given by Equation (1).

Let us define $b = \pm\gamma/m$ and $c = k/m = \omega_o^2$, where ω_o is the characteristic frequency of the oscillators, and write Equations (2) and (3) as one equation of motion

$$\ddot{x}(t) + b\dot{x}(t) + \omega_o^2 x(t) = 0 , \tag{4}$$

with the understanding that the damped and amplified oscillators require $b > 0$ and $b < 0$, respectively, and that the variable $x(t)$ describes either damped or amplified oscillator. Let $\hat{D} = d^2/dt^2 + bd/dt + \omega_o^2$ be a linear operator, then Equation (4) can be written in the compact form $\hat{D}x(t) = 0$.

In the following, we derive the SLs, NLs and GFs for the Bateman oscillators whose equations of motion are $\hat{D}x(t) = 0$.

2.2. Novel Method to Derive Standard and Null Lagrangians

The first-derivative term in Equation (4) can be removed using the standard transformation of the dependent variable [17]. The transformation is

$$x(t) = x_1(t)e^{-bt/2} , \tag{5}$$

where $x_1(t)$ is the transformed dependent variable, and it gives

$$\ddot{x}_1(t) + \left(\omega_o^2 - \frac{1}{4}b^2\right)x_1(t) = 0 . \tag{6}$$

Despite the fact that the first derivative term is removed, the coefficient b is still present in the transformed equation of motion. However, if $b = 0$, then $x_1(t) = x(t)$ and Equation (6) becomes the equation of motion for a undamped harmonic oscillator [18,19].

The standard Lagrangian for this equation is

$$L_s[\dot{x}_1(t), x_1(t)] = \frac{1}{2}\left[(\dot{x}_1(t))^2 - \left(\omega_o^2 - \frac{1}{4}b^2\right)x_1^2(t)\right], \quad (7)$$

and its substitution into the EL equation gives Equation (6). Let us now follow [20,21] and consider the following Lagrangian

$$L_n[\dot{x}_1(t), x_1(t), t] = C_1\dot{x}_1(t)x_1(t) + C_2[\dot{x}_1(t)t + x_1(t)] + C_4\dot{x}_1(t) + C_6\,, \quad (8)$$

where C_1, C_2, C_4 and C_6 are constants. It is easy to verify that $L_n[\dot{x}_1(t), x_1(t), t]$ is the null Lagrangian with the constants being arbitrary, and that this NL is constructed to the lowest order of its dynamical variables [20,21]. The NL can be added to $L_s[\dot{x}_1(t), x_1(t)]$ without changing the form of the equation of motion (see Equation (6)) resulting from it.

Since the original equations of motion for the Bateman oscillators depend on the dynamical variable $x(t)$ not $x_1(t)$ (see Equations (2) and (3)), we now use the inverse transform given by Equation (5) to convert the variable $x_1(t)$ into $x(t)$ in both $L_s[\dot{x}_1(t), x_1(t)]$ and $L_n[\dot{x}_1(t), x_1(t), t]$. The resulting Lagrangian is

$$L[\dot{x}(t), x(t), t] = L_{CK}[\dot{x}(t), x(t), t] + L_n[\dot{x}(t), x(t), t]\,, \quad (9)$$

where

$$L_{CK}[\dot{x}(t), x(t), t] = \frac{1}{2}\left[(\dot{x}(t))^2 - \omega_o^2 x^2(t)\right]e^{bt}\,, \quad (10)$$

is the CK Lagrangian [7,8], derived here independently; comparison of the CK and Bateman Lagrangians (Equations (1) and (10)) shows significant differences between them. The presented method gives the following null Lagrangian

$$L_n[\dot{x}(t), x(t), t] = \sum_{i=1}^{3} L_{ni}[\dot{x}(t), x(t), t]\,, \quad (11)$$

where the partial null Lagrangians are

$$L_{n1}[\dot{x}(t), x(t), t] = \left(C_1 + \frac{1}{2}b\right)\left[\dot{x}(t) + \frac{1}{2}bx(t)\right]x(t)e^{bt}\,, \quad (12)$$

$$L_{n2}[\dot{x}(t), x(t), t] = C_2\left[\left(\dot{x}(t) + \frac{1}{2}bx(t)\right)t + x(t)\right]e^{bt/2}\,, \quad (13)$$

and

$$L_{n3}[\dot{x}(t), x(t), t] = C_4\left[\dot{x}(t) + \frac{1}{2}bx(t)\right]e^{bt/2} + C_6\,, \quad (14)$$

where C_1, C_2, C_4 and C_6 are arbitrary but their physical units are different such that all partial null Lagrangians have the same units of energy. These are new null Lagrangians for the Bateman oscillators. The fact that $L_n[\dot{x}(t), x(t), t]$ and its partial Lagrangians are the NLs can be shown by defining $\hat{EL}$ to be the differential operator of the EL equation and verifying that $\hat{EL}(L_n) = 0$ as well as $\hat{EL}(L_{ni}) = 0$. It must be also noted that $b = 0$ reduces $L_n[\dot{x}(t), x(t), t]$ to the previously obtained null Lagrangian [15].

2.3. Gauge Functions and General Null Lagrangians

For each partial null Lagrangian, its corresponding partial gauge function can be obtained and the results are

$$\Phi_{n1}[x(t), t] = \frac{1}{2}\left(C_1 + \frac{1}{2}b\right)x^2(t)e^{bt}\,, \quad (15)$$

$$\Phi_{n2}[x(t), t] = C_2 x(t)te^{bt/2}\,, \quad (16)$$

and

$$\phi_{n3}[x(t), t] = C_4 x(t) e^{bt/2} + C_6 t \,. \tag{17}$$

These partial gauge functions can be added together to form the gauge function

$$\phi_n[x(t), t] = \sum_{i=1}^{3} \phi_{ni}[x(t), t] \,. \tag{18}$$

The derived gauge function and partial gauge functions reduce to those previously obtained [15] when $b = 0$ is assumed.

The above partial gauge functions are obtained to the lowest order of the dynamical variables for the Bateman oscillators. Therefore, the only way to generalize the gauge functions given by Equations (15)–(17), without changing the order of their dynamical variables, is to replace the constants C_1, C_2, C_4 and C_6 with the corresponding functions $f_1(t)$, $f_2(t)$, $f_4(t)$ and $f_6(t)$, which must be continuous and at least twice differentiable, and must depend only on the independent variable t. To obey the assumption that our method of constructing null Lagrangians is limited to the lowest order of the dynamical variables, the functions cannot depend on the space coordinates. An interesting result is that the functions give additional degrees of freedom in constructing the null Lagrangians and their gauge functions.

We follow [20,21] and call these GFs the general GFs and write them here as

$$\phi_{gn1}[x(t), t] = \frac{1}{2}\left[f_1(t) + \frac{1}{2}b\right] x^2(t) e^{bt} \,, \tag{19}$$

$$\phi_{gn2}[x(t), t] = f_2(t) x(t) t e^{bt/2} \,, \tag{20}$$

and

$$\phi_{gn3}[x(t), t] = f_4(t) x(t) e^{bt/2} + f_6(t) t \,. \tag{21}$$

The general gauge function is obtained by adding these partial gauge functions

$$\phi_{gn}[x(t), t] = \sum_{i=1}^{3} \phi_{gni}[x(t), t] \,. \tag{22}$$

The results given by Equation (19) through (22) are new general gauge functions for the Bateman oscillators and they generalize those previously obtained for Newton's law of inertia [20] and for linear undamped oscillators [21] for which $b = 0$.

Using the general gauge function $\phi_{gn}[x(t), t]$, the resulting general null Lagrangian can be calculated and the result is

$$L_{gn}[\dot{x}(t), x(t), t] = \sum_{i=1}^{3} L_{gni}[\dot{x}(t), x(t), t] \,, \tag{23}$$

where the partial null Lagrangians are given by

$$\begin{aligned} L_{gn1}[\dot{x}(t), x(t), t] = {} & \left[f_1(t) + \frac{1}{2}b\right]\left[\dot{x}(t) + \frac{1}{2}bx(t)\right] x(t) e^{bt} \\ & + \frac{1}{2}\dot{f}_1(t) x^2(t) e^{bt} \,, \end{aligned} \tag{24}$$

$$\begin{aligned} L_{gn2}[\dot{x}(t), x(t), t] = {} & \left[f_2(t)\dot{x}(t) + \dot{f}_2(t) x(t)\right] t e^{bt/2} \\ & + \left(1 + \frac{1}{2}bt\right) f_2(t) x(t) e^{bt/2} \,, \end{aligned} \tag{25}$$

and

$$L_{gn3}[\dot{x}(t), x(t), t] = \left[f_4(t)\dot{x}(t) + \dot{f}_4(t)x(t)\right]e^{bt/2} + \frac{1}{2}bf_4(t)x(t)e^{bt/2} + \left[\dot{f}_6(t)t + f_6(t)\right] . \tag{26}$$

The obtained results show that the replacement of the constants C_1, C_2, C_4 and C_6 by the functions $f_1(t)$, $f_2(t)$, $f_4(t)$ and $f_6(t)$ gives the Lagrangian $L_{gn}[\dot{x}(t), x(t), t]$, which is the null Lagrangian that significantly generalizes $L_n[\dot{x}(t), x(t), t]$ given by Equation (11). From a physical point of view, it is required that the functions have correct physical units, so each partial general Lagrangian has the units of energy. The functions may also be further constrained by postulating the invariance of action for the null Lagrangians and introducing the so-called exact gauge functions [20], which allows specifying the end points of some of these functions.

The general null Lagrangian $L_{gn}[\dot{x}(t), x(t), t]$ and its partial null Lagrangians reduce to those previously found [20,21] when $b = 0$. These Lagrangians depend on four functions, which can be constrainted by the initial conditions if they are specified. The initial conditions are used to make the action invariant, which requires that the gauge functions become zero at the initial conditions. This constrains the values of the four functions at the initial conditions, and the gauge functions that make the action invariant are called the exact gauge functions [20].

Since the null Lagrangians do not affect the derivation of the equations of motion, their existence is not restricted by the Helmholtz conditions [6,16]. However, the conditions seem to imply that the Caldirola-Kanai Lagrangian cannot be constructed for the Bateman oscillators. In the following, we consider and discuss this problem in detail.

2.4. Role of the Caldirola-Kanai Lagrangian

The Caldirola-Kanai Lagrangian, $L_{CK}[\dot{x}(t), x(t), t]$, when substituted into the EL equation, yields $[\hat{D}x(t)]e^{bt} = 0$, which is consistent with all Helmholtz conditions that are valid for any system of ordinary differential equations [16]. However, with $e^{bt} \neq 0$, the resulting $\hat{D}x(t) = 0$ does obey the first and second Helmholtz conditions but fails to satisfy the third condition. This shows that after the division by e^{bt} the equation of motion fails to satisfy the third condition [22]; see further discussion below Equation (30).

Since the CK Lagrangian depends explicitly on time, the energy function [18,19] must be calculated. Let $E_{CK}[\dot{x}(t), x(t), t]$ be the energy function for $L_{CK}[\dot{x}(t), x(t), t]$ given by

$$E_{CK}[\dot{x}(t), x(t), t] = \dot{x}(t)p_c(t) - L_{CK}[\dot{x}(t), x(t), t] , \tag{27}$$

where the canonical momentum $p_c(t)$ is

$$p(t) = \frac{\partial L_{CK}[\dot{x}(t), x(t), t]}{\partial \dot{x}(t)} = \dot{x}(t)e^{bt} , \tag{28}$$

and is different than the linear momentum $p(t) = \dot{x}(t)$.

Then, the energy function $E_{CK}[\dot{x}(t), x(t), t]$ can be written as

$$E_{CK}[\dot{x}(t), x(t), t] = E_{tot}[\dot{x}(t), x(t)]e^{bt} . \tag{29}$$

where the total energy is

$$E_{tot}[\dot{x}(t), x(t)] = \frac{1}{2}\left[(\dot{x}(t))^2 + \omega_o^2 x^2(t)\right] . \tag{30}$$

According to Equations (29) and (30), the energy function depends explicitly on time through its exponential term; moreover, both the energy function and total energy do depend on time through $x(t)$.

Having obtained $E_{CK}[\dot{x}(t), x(t), t]$, we use

$$\frac{dE_{CK}[\dot{x}(t), x(t), t]}{dt} = -\frac{\partial L_{CK}[\dot{x}(t), x(t), t]}{\partial t} , \tag{31}$$

and find the following equation of motion

$$[\hat{D}x(t)]\dot{x}(t)e^{bt} = 0 . \tag{32}$$

Since $\dot{x}(t)e^{bt} \neq 0$, Equation (32) becomes $\hat{D}x(t) = 0$, which is the equation of motion for the Bateman oscillators (see Equation (4)) with the well-known solutions for $x(t)$ [18] that decrease (increase) with time as $e^{-bt/2}$ for the damped (amplified) oscillators.

The exponentially decreasing (increasing) in time solutions to $\hat{D}x(t) = 0$ make $E_{CK}[\dot{x}(t), x(t), t]$ = const, when substituted into Equation (29) because of the cancellation of e^{-bt} and e^{bt}, and show that $E_{tot}[\dot{x}(t), x(t)]$ decreases (increases) exponentially in time for the damped (amplified) oscillators. The fact that $E_{tot}[\dot{x}(t), x(t)] \neq$ const but $E_{CK}[\dot{x}(t), x(t), t] = E_{tot}[\dot{x}(t), x(t)]e^{bt}$ = const guarantees that the equation of motion $[\hat{D}x(t)]e^{bt} = 0$ resulting from the CK Lagrangian satisfies the third Helmholtz condition; however, the equation $[\hat{D}x(t)] = 0$ does not satisfy it as is already pointed out at the begining of this section.

The role of the CK Lagrangian in deriving the equation of motion for the Bateman oscillators has been questioned in the literature [23] based on the previous work [24]. The main conclusion of that previous research was that the CK Lagrangian does not describe the Bateman oscillators but instead a different oscillatory system with its mass increasing ($b > 0$) or decreasing ($b < 0$) exponentially in time. However, as pointed out recently in [25], the previous work has some conceptual errors that led to incorrect conclusions.

The results presented in this paper are consistent with those given in [25] as we demonstrated that the energy function (see Equation (27)) that depends on the canonical momentum $p_c(t)$ is constant in time, and that the total energy (see Equation (30)) that depends on the linear momentum $p(t)$ is not constant in time. This shows that the total energy and the linear momentum decrease (increase) in time for the damped (amplified) Bateman oscillators, which is consistent with the physical picture of these dynamical systems. The increase (decrease) of the canonical momentum in time does not contradict this picture but instead guarantees that the CK Lagrangian can be used to derive the equations of motion for the Bateman oscillators.

3. From Undriven to Driven Bateman Oscillators

3.1. Total Energy Function

Having demonstrated the validity of the CK Lagrangian for the Bateman oscillators, we now investigate effects of the gauge functions on the CK Lagrangian. Let us combine the CK Lagrangian, given by Equation (10), and the general null Lagrangian, given by Equation (23) through (26), together to obtain the following Lagrangian $L[\dot{x}(t), x(t), t] = L_{CK}[\dot{x}(t), x(t), t] + L_{gn}[\dot{x}(t), x(t), t]$. Using this new Lagrangian, we find the resulting energy function to be

$$E[\dot{x}(t), x(t), t] = \frac{1}{2}\left[(\dot{x}(t))^2 + \left(\omega_o^2 - \bar{\omega}_o^2(t)\right)x^2(t)\right]e^{bt} \\ - F(t)x(t)e^{bt/2} - G(t) , \tag{33}$$

where

$$\bar{\omega}_o^2(t) = \left[\left(f_1(t) + \frac{1}{2}b\right)b + \dot{f}_1(t)\right] , \tag{34}$$

$$F(t) = \left[\left(1 + \frac{1}{2}bt\right)f_2(t) + \dot{f}_2(t)t + \dot{f}_4(t) + \frac{1}{2}bf_4(t)\right] , \tag{35}$$

and

$$G(t) = \dot{f}_6(t)t + f_6(t) \,. \tag{36}$$

We make the frequency shift $\bar{\omega}_o^2$ time-independent by taking $f_1(t) = C_1$ and obtaining $\bar{\omega}_o^2 = (C_1 + b/2)b$. We also define the total energy

$$E_{tot}[\dot{x}(t), x(t)] = \frac{1}{2}\left[(\dot{x}(t))^2 + \omega_s^2 x^2(t)\right] , \tag{37}$$

where $\omega_s^2 = \omega_o^2 - \bar{\omega}_o^2$ = const. It must be noted that $E_{tot}[\dot{x}(t), x(t)]$ is not constant because $x(t)$ and $\dot{x}(t)$ decrease exponentially in time if $b > 0$ or increase exponentially in time if $b < 0$.

Using Equation (37), we write Equation (33) as

$$E[\dot{x}(t), x(t), t] = \left[E_{tot}[\dot{x}(t), x(t)] - F(t)x(t)e^{-bt/2} - G(t)e^{-bt}\right]e^{bt} , \tag{38}$$

which shows that the total energy of the system is modified by the presence of $F(t)$ and $G(t)$ that depend on the functions $f_2(t)$, $f_4(t)$ and $f_6(t)$, which are arbitrary. Therefore, these arbitrary functions can be chosen so that the difference between $E_{tot}[\dot{x}(t), x(t)]$ and the two energy terms $F(t)x(t)e^{-bt/2}$ and $G(t)e^{-bt}$ approach zero as $t \to \infty$.

It must be also noted that in the special case of $b = 0$, Equation (38) reduces to Equation (40) derived in [20]. However, if $b = 0$ and $f_2(t) = C_2$, $f_4(t) = C_4$ and $f_6(t) = C_6$, then the total energy function is equal to that obtained in [20].

3.2. Gauge Functions and Forces

Despite the fact that the partial null Lagrangians were used in deriving the total energy functions, these null Lagrangians were derived from the partial gauge functions given by Equation (19) through (21). Based on definitions given by Equation (34) through (36), it can be seen that the partial gauge function $\phi_{gn1}[x(t), t]$ contributes only to $\bar{\omega}_o^2(t)$, and that $F(t)$ is determined by $\phi_{gn2}[x(t), t]$ and also by $\phi_{gn3}[x(t), t]$, which contributes partially to both $F(t)$ and $G(t)$.

Since the energy $E[\dot{x}(t), x(t), t]$ is modified by the presence of the extra energies $F(t)x(t)$ and $G(t)$, a new Lagrangian corresponding to $E[\dot{x}(t), x(t), t]$ must also be modified, which is achieved by adding these extra terms to $L_{CK}[\dot{x}(t), x(t), t]$ given by Equation (10). Then, the modified CK Lagrangian $L_{CK,mod}[\dot{x}(t), x(t), t]$ becomes

$$\begin{aligned} L_{CK,mod}[\dot{x}(t), x(t), t] = \frac{1}{2}\left[(\dot{x}(t))^2 - \omega_s^2 x^2(t)\right]e^{bt} \\ - F(t)x(t)e^{bt/2} - G(t) \,, \end{aligned} \tag{39}$$

which is the well-known Lagrangian for forced and damped oscillators [18,19]. Our results demonstrate that the gauge function is responsible for introducing two extra energy terms to the original CK Lagrangian.

Using $L_{CK,mod}[\dot{x}(t), x(t), t]$, the following equation of motion is obtained

$$[\ddot{x}(t) + b\dot{x}(t) + (\omega_o^2 - \bar{\omega}_o^2(t))x(t)]e^{bt} = F(t)e^{bt/2} , \tag{40}$$

which can also be written as

$$\ddot{x}(t) + b\dot{x}(t) + \omega_s^2 x(t) = F(t)e^{-bt/2} \,. \tag{41}$$

An interesting result is that the solutions to the homogeneous equation of Equation (41) depend on $e^{-bt/2}$, which is the exponential factor as in the forcing function; see below for the full complemantary and particular solutions.

The definition of $F(t)$ shows that it is determined by the gauge function $\phi_{gn2}[x(t), t]$, which contributes through its function $f_2(t)$ and its derivative, and also partially by

$\phi_{gn3}[x(t), t]$, which contributes through its function $f_4(t)$ and its derivative. Moreover, $f_2(t)$ and $f_4(t)$ can be any functions of t as long as they are differentiable. More constraints on these functions can be imposed after invariance of the action is considered [20] or the initial conditions are specified.

Let us define $\beta = b/2$ and write Equation (40) as

$$\ddot{x}(t) + 2\beta\dot{x}(t) + \omega_s^2 x(t) = F(t)e^{-\beta t} . \tag{42}$$

Since β = const and ω_s = const, the solution to the homogeneous part of Equation (41) is well-known [18,19] and given by

$$x_h(t) = x_o e^{(-\beta + i\omega)t} , \tag{43}$$

where x_o is an integration constant and $\omega = \sqrt{\omega_s^2 - \beta^2}$ is the natural frequency of oscillations.

To find the particular solution $x_p(t)$, the force function $F(t)$ must be specified. Let us take

$$F(t) = F_o e^{(i\Omega)t} , \tag{44}$$

where F_o and Ω are the constant amplitude and frequency of the driving force, respectively. Then, we seek the particular solution in the following form

$$x_p(t) = \Gamma e^{(-\beta + i\Omega)t} , \tag{45}$$

where Γ is a constant to be determined. Substituting Equation (45) into Equation (42) and using Equation (44), we obtain

$$\Gamma = \frac{F_o}{\omega^2 - \Omega^2} . \tag{46}$$

Thus, the solution to Equation (41) is

$$x(t) = x_o e^{(-\beta + i\omega)t} + \frac{F_o}{\omega^2 - \Omega^2} e^{(-\beta + i\Omega)t} . \tag{47}$$

The main difference between our solution for $x(t)$ and that found in textbooks of Classical Mechanics [18,24] is that both $x_h(t)$ and $x_p(t)$ decay exponentially in time, which results in a different Γ that shows a resonance if $\omega = \Omega$.

To derive the standard solution given in the textbooks [18,24], it is required that

$$F(t) = F_o e^{(\beta + i\Omega)t} . \tag{48}$$

This is allowed as $F(t)$ is arbitrary as long as it is differentiable. However, $F(t)$ is expressed in terms of the functions $f_2(t)$ and $f_4(t)$ and their derivatives, these two functions are still to be determined once $F(t)$ is specified. Since $f_2(t)$ and $f_4(t)$ are arbitrary, we may take either $f_2(t) = C_2$ = const and solve a first-order ordinary differential equation (ODE) for $f_4(t)$ or assume that $f_4(t) = C_4$ = const and solve another ODE for $f_2(t)$.

The important points of the procedure described above are: (i) Any null Lagrangian has no effect on the equation of motion; (ii) Null Lagrangians depend explicitly on time, so when added to the standard Lagrangian they make the total Lagrangian depend on time; (iii) The time-dependent Lagrangian requires that the energy function is calculated; (iv) The resulting energy function has extra energy-like terms that are not null Lagrangians; (v) By adding these energy-like terms to the standard Lagrangian, the derived equation of motion becomes inhomogeneous (driven). This shows an interesting connection between the null Lagrangians and their gauge functions, and classical forces [20,21].

The presented results demonstrate that the CK Lagrangian modified by the gauge functions gives the equations of motion for the driven Bateman oscillators, which shows that the gauge functions can be used to introduce forces to undriven dynamical systems [21].

The result justifies searches for null Lagrangians and their gauge functions. Our results concern only the Bateman oscillators but the same method can be applied to other dynamical systems, whose existing standard Lagrangians may give hints how to construct null Lagrangians for these systems. The role of null Lagrangians in classical physics has not yet been established, with [26,27] being the exceptions.

4. Conclusions

We developed the Lagrangian formalism for Bateman oscillators and used it to derive the Caldirola-Kanai (CK) and null Lagrangians. For each null Lagrangian the corresponding gauge function was obtained. Our results confirm the previous findings [7,8,25] that the CK Lagrangian can be used to derive the equations of motion for the Bateman oscillators. We also considered the Helmholtz conditions that guarantee the existence of Lagrangians for given ODEs, and demonstrated that the energy function obtained from the CK Lagrangian is the sufficient condition for the resulting equation of motion to satisfy the conditions.

The results obtained in this paper contradict the previous claims [23,24] that the CK Lagrangian is not valid for the Bateman oscillators but instead it represents a different dynamical system in which mass increases or decreases exponentially in time. As a result of this contradiction, derivations of the Kanai-Caldirola propagator in the de Broglie-Bohm theory [28], which are based on the CK Lagrangian with its mass being time-dependent [29], must be taken with caution.

Our main result is that the CK Lagrangian modified by the presence of extra energy terms generated by the partial gauge functions can be used to describe the driven Bateman oscillators. The modified CK Lagrangian given by Equation (39) and the resulting equation of motion for the driven Bateman oscillators (see Equation (41)) clearly show that the Lagrangian formulation is possible, and that the undriven Bateman oscillators can be converted into the driven ones if the gauge functions are taken into account.

The Pais-Uhlenbeck [30] and Feshbach-Tikochinsky [31] methods to quantize the Bateman oscillators are based on the Bateman Lagrangian given by Equation (1). Using these methods, it was shown that quantization of the damped quantum harmonic oscillator can be done in terms of ladder operators [32], which is partially incorrect due to a no-go theorem [33]. Moreover, there seems to be also a quantum vacuum problem in [32] as demonstrated recently [34]. Quantization of the damped harmonic oscillator can also be performed using the CK Lagrangian given by Equation (10), whose symmetries were investigated and constant-of-motion operators that generate the Heisenberg-Weyl algebra were introduced [35]. Based on the results obtained in this paper, it may also be possible to quantize the driven Bateman oscillators using the modified CK Lagrangian given by Equation (39), which is suggested for future work.

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