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New Generalized Hermite–Hadamard–Mercer's Type Inequalities Using (k, ψ)-Proportional Fractional Integral Operator

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Abstract: In this paper, by using Jensen–Mercer's inequality we obtain Hermite–Hadamard–Mercer's type inequalities for a convex function employing left-sided (k, ψ)-proportional fractional integral operators involving continuous strictly increasing function. Our findings are a generalization of some results that existed in the literature.

Keywords: convex function; Jensen–Mercer's inequality; Hermite–Hadamard–Mercer's inequality; (k, ψ)-proportional fractional integral operator

MSC: 26A33; 26A51; 26D10; 26D15

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Citation: Desta, H.D.; Nwaeze, E.R.; Abdi, T.; Mijena, J.B. New Generalized Hermite–Hadamard–Mercer's Type Inequalities Using (k, ψ)-Proportional Fractional Integral Operator. *Foundations* **2023**, *3*, 49–62. https://doi.org/10.3390/foundations3010005

Academic Editors: Sotiris K. Ntouyas and Wei-Shih Du

Received: 18 October 2022
Revised: 17 November 2022
Accepted: 3 January 2023
Published: 11 January 2023

1. Introduction

Convex functions play an important role in applied and pure mathematics. Several investigation has been carried out to extend and generalize results on inequalities for convex functions. Among the monumental inequalities which holds for convex functions, Hermite–Hadamard–Mercer's inequality is one of them. Therefore, our focus is on this inequality to make a new generalization by using left-sided (k, ψ)-proportional fractional integral operators involving continuous strictly increasing functions.

Definition 1. *A function $\mathcal{F} : [\mu_1, \mu_2] \to \mathbb{R}$ is said to be a convex function if the inequality*

$$\mathcal{F}\Big(\xi\lambda_1 + (1-\xi)\lambda_2\Big) \leq \xi\mathcal{F}(\lambda_1) + (1-\xi)\mathcal{F}(\lambda_2), \tag{1}$$

holds for all $\lambda_1, \lambda_2 \in [\mu_1, \mu_2]$ and $\xi \in [0, 1]$. We say that $\mathcal{F}$ is a concave function if the inequality (1) is reversed.

For any convex function $\mathcal{F}$ on $[\mu_1, \mu_2]$ the following inequality holds

$$\mathcal{F}\left(\frac{\mu_1 + \mu_2}{2}\right) \leq \frac{1}{\mu_2 - \mu_1}\int_{\mu_1}^{\mu_2} \mathcal{F}(x)dx \leq \frac{\mathcal{F}(\mu_1) + \mathcal{F}(\mu_2)}{2}, \tag{2}$$

which is called the Hermite–Hadamard inequality and it is one of the most commonly used inequalities for many researchers in the field of inequality theory. It was first discovered by Hermite in 1881 [1]. However, this result was nowhere mentioned in the literature until it was augmented by Hadamard in 1893 [2]. For more generalizations of Hermite–Hadamard inequality (2) with different kind of convexity see for instance [3–6] and references cited therein. The well-known Jensen inequality in [7] states that if $\mathcal{F}$ is convex function on

$[\mu_1, \mu_2]$ and for all $y_i \in [\mu_1, \mu_2]$ and for any $\xi_i \in [0,1]; i = 1, 2, ..., n$ with $\sum_{i=1}^{n} \xi_i = 1$ we have

$$\mathcal{F}\left(\sum_{i=1}^{n} \xi_i y_i\right) \leq \sum_{i=1}^{n} \xi_i \mathcal{F}(y_i). \tag{3}$$

Lemma 1 ([8])**.** *Suppose that a function $\mathcal{F} : [\mu_1, \mu_2] \to \mathbb{R}$ is a convex function on $[\mu_1, \mu_2]$. Then, we have*

$$\mathcal{F}(\mu_1 + \mu_2 - y) \leq \mathcal{F}(\mu_1) + \mathcal{F}(\mu_2) - \mathcal{F}(y). \tag{4}$$

In [8], Mercer proved the variant of Jensen inequality known as the Jensen–Mercer's inequality:

Theorem 1. *If $\mathcal{F}$ is a convex function on $[\mu_1, \mu_2]$, then*

$$\mathcal{F}\left(\mu_1 + \mu_2 - \sum_{i=1}^{n} \xi_i y_i\right) \leq \mathcal{F}(\mu_1) + \mathcal{F}(\mu_2) - \sum_{i=1}^{n} \xi_i \mathcal{F}(y_i),$$

for each $y_i \in [\mu_1, \mu_2]$ and for any $\xi_i \in [0,1]; i = 1, 2, ..., n$ with $\sum_{i=1}^{n} \xi_i = 1$.

Recently, many scholars have given attention and established the inequality of Hermite–Hadamard–Mercer's inequalities using Riemann–Liouville fractional integrals, the conformable fractional integral, Katugampola fractional integrals, generalized Riemann–Liouville integrals and so on. For instance, see [9–13] and the cited reference therein. Therefore, our motive is to study about Hermite–Hadamard–Mercer's type inequalities by employing the (k, ψ)-proportional fractional integral operator involving continuous strictly increasing functions to generate a new generalized inequalities.

The paper is organized as follows: In Section 2, we give some definition and results which we will use through out this paper. In Section 3, we state our main result on Hermite–Hadamard–Mercer's type inequalities. Finally, Section 4 is devoted to the conclusion of our work. Section 1 is devoted to the introduction and motivation of our proposed study.

2. Preliminaries

In this section, we present some basic definitions and properties of fractional integral operators which helps us to obtain our new results.

Definition 2 ([14])**.** *Suppose that the function $\mathcal{F}$ is integrable on $[\mu_1, \mu_2]$ and $\mu_1 \geq 0$. Then, for all $\beta > 0$, the right-and left-sided Riemann–Liouville fractional integral of a function $\mathcal{F}$ of order β are, respectively, given by*

$$\mathcal{J}^{\beta}_{\mu_1^+} \mathcal{F}(y) = \frac{1}{\Gamma(\beta)} \int_{\mu_1}^{y} (y - u)^{\beta - 1} \mathcal{F}(u) du, y > \mu_1, \tag{5}$$

and

$$\mathcal{J}^{\beta}_{\mu_2^-} \mathcal{F}(y) = \frac{1}{\Gamma(\beta)} \int_{y}^{\mu_2} (u - y)^{\beta - 1} \mathcal{F}(u) du, y < \mu_2, \tag{6}$$

where $\Gamma(\beta) = \int_0^\infty e^{-x} x^{\beta-1} dx$ is the gamma function. The notations $\mathcal{J}^{\beta}_{\mu_1^+} \mathcal{F}(y)$ and $\mathcal{J}^{\beta}_{\mu_2^-} \mathcal{F}(y)$ are, respectively, called the left- and right-sided Riemann–Liouville fractional integral of a function $\mathcal{F}$ of order β.

Definition 3 ([14,15]). *Suppose that the function $\mathcal{F}$ is integrable on the interval $[\mu_1, \mu_2]$ and let ψ be an increasing function, where $\psi(y) \in C^1([\mu_1, \mu_2], \mathbb{R})$ such that $\psi'(y) \neq 0, y \in [\mu_1, \mu_2]$. Then, for all $\beta > 0$ we have*

$$\left({}^{\psi}\mathcal{J}^{\beta}_{\mu_1^+} \mathcal{F} \right)(y) = \frac{1}{\Gamma(\beta)} \int_{\mu_1}^{y} \psi'(u) \Big[\psi(y) - \psi(u) \Big]^{\beta-1} \mathcal{F}(u) du, y > \mu_1, \tag{7}$$

and

$$\left({}^{\psi}\mathcal{J}^{\beta}_{\mu_2^-} \mathcal{F} \right)(y) = \frac{1}{\Gamma(\beta)} \int_{y}^{\mu_2} \psi'(u) \Big[\psi(u) - \psi(y) \Big]^{\beta-1} \mathcal{F}(u) du, y < \mu_2. \tag{8}$$

The notations $\left({}^{\psi}\mathcal{J}^{\beta}_{\mu_1^+} \mathcal{F} \right)(y)$ and $\left({}^{\psi}\mathcal{J}^{\beta}_{\mu_2^-} \mathcal{F} \right)(y)$ are, respectively, called the left- and right-sided ψ-Riemann–Liouville fractional integrals of a function $\mathcal{F}$ of order β.

Definition 4 ([16]). *For $\delta \in [0, 1]$, let the function $\Upsilon_0, \Upsilon_1 : [0, 1] \times \mathbb{R} \to [0, \infty)$ be continuous for all $\omega \in \mathbb{R}$, we have*

$$\lim_{\delta \to 0^+} \Upsilon_1(\delta, \omega) = 1, \lim_{\delta \to 0^+} \Upsilon_0(\delta, \omega) = 0, \lim_{\delta \to 1^-} \Upsilon_1(\delta, \omega) = 0, \lim_{\delta \to 1^-} \Upsilon_0(\delta, \omega) = 1, \tag{9}$$

and $\Upsilon_1(\delta, \omega) \neq 0, \delta \in [0, 1], \Upsilon_0(\delta, \omega) \neq 0, \delta \in (0, 1]$. Then, the proportional fractional derivative operator of order δ is defined by

$$D^{\delta} f(\omega) = \Upsilon_1(\delta, \omega) \mathcal{F}(\omega) + \Upsilon_0(\delta, \omega) \mathcal{F}'(\omega). \tag{10}$$

In the case when $\Upsilon_1(\delta, \omega) = (1 - \delta)$ and $\Upsilon_0(\delta, \omega) = \delta$ the Equation (10) becomes

$$D^{\delta} \mathcal{F}(\omega) = (1 - \delta) \mathcal{F}(\omega) + \delta \mathcal{F}'(\omega).$$

Note that $\lim_{\delta \to 0^+} D^{\delta} \mathcal{F}(\omega) = \mathcal{F}(\omega)$ and $\lim_{\delta \to 1^-} D^{\delta} \mathcal{F}(\omega) = \mathcal{F}'(\omega)$. The related generalized proportional fractional integrals are defined as follows:

Definition 5 ([16]). *Suppose that the function $\mathcal{F}$ is integrable on $[\mu_1, \mu_2]$, let $\delta \in (0, 1]$, we have for all $\beta \in \mathbb{C}, Re(\beta) \geq 0$*

$$\left(\mathcal{J}^{\beta,\delta}_{\mu_1^+} \mathcal{F} \right)(y) = \frac{1}{\delta^{\beta} \Gamma(\beta)} \int_{\mu_1}^{y} (y - u)^{\beta-1} \mathcal{F}(u) du, y > \mu_1, \tag{11}$$

and

$$\left(\mathcal{J}^{\beta,\delta}_{\mu_2^-} \mathcal{F} \right)(y) = \frac{1}{\delta^{\beta} \Gamma(\beta)} \int_{y}^{\mu_2} (u - y)^{\beta-1} \mathcal{F}(u) du, y < \mu_2. \tag{12}$$

The notations $\left(\mathcal{J}^{\beta,\delta}_{\mu_1^+} \mathcal{F} \right)(y)$ and $\left(\mathcal{J}^{\beta,\delta}_{\mu_2^-} \mathcal{F} \right)(y)$ are called, respectively, left- and right-sided generalized proportional fractional integral operators of order β.

Remark 1. *If $\delta = 1$ in Definition 5, then we obtain the classical Riemann–Liouville fractional integral.*

Definition 6 ([16]). *For the integrable function $\mathcal{F}$, let $\delta > 0, m \in \mathbb{N}$ such that $m = [Re(\beta)] + 1$, we have for all $\beta \in \mathbb{C}, Re(\beta) \geq 0$,*

$$\begin{aligned} \left(D^{\beta,\delta}_{\mu_1^+} \mathcal{F} \right)(y) &= D^{m,\delta} \mathcal{J}^{m-\beta,\delta}_{\mu_1^+} \mathcal{F}(y) \\ &= \frac{D^{m,\delta}_{y}}{\delta^{m-\beta} \Gamma(m - \beta)} \int_{\mu_1}^{y} exp\left[\frac{\delta - 1}{\delta} (y - u) \right] (y - u)^{m-\beta-1} \mathcal{F}(u) du, \end{aligned} \tag{13}$$

and

$$\left(D^{\beta,\delta}_{\mu_2^-}\mathcal{F}\right)(y) = {}_{\gamma}D^{m,\delta}\mathcal{J}^{m-\beta,\delta}_{\mu_2^-}\mathcal{F}(y) \tag{14}$$
$$= \frac{{}_{\gamma}D^{m,\delta}_{y}}{\delta^{m-\beta}\Gamma(m-\beta)}\int_y^{\mu_2} exp\left[\frac{\delta-1}{\delta}(u-y)\right](u-y)^{m-\beta-1}\mathcal{F}(u)du,$$

where

$$D^{m,\delta} = \underbrace{D^{\delta}D^{\delta}...D^{\delta}}_{m-times} \tag{15}$$

and

$$\left({}_{\gamma}D^{\delta}\mathcal{F}\right)(y) = (1-\delta)\mathcal{F}(y) - \delta\mathcal{F}'(y), {}_{\gamma}D^{m,\delta} = \underbrace{{}_{\gamma}D^{\delta}{}_{\gamma}D^{\delta}...{}_{\gamma}D^{\delta}}_{m-times}. \tag{16}$$

The notations $\left(D^{\beta,\delta}_{\mu_1^+}\mathcal{F}\right)(y)$ and $\left(D^{\beta,\delta}_{\mu_2^-}\mathcal{F}\right)(y)$ are, respectively, called the left- and right-sided proportional fractional derivatives of a function $\mathcal{F}$ of order β.

Definition 7 ([17])**.** *For an integrable function $\mathcal{F}$ on the interval $[\mu_1, \mu_2]$ and for $\psi \in C^1([\mu_1, \mu_2], \mathbb{R})$, such that $\psi'(y) \neq 0, y \in [\mu_1, \mu_2]$. We have for all $\beta, k > 0$*

$$\left({}^{(k,\psi)}\mathcal{J}^{\beta}_{\mu_1^+}\mathcal{F}\right)(y) = \frac{1}{k\Gamma_k(\beta)}\int_{\mu_1}^{y}\psi'(u)\Big[\psi(y)-\psi(u)\Big]^{\frac{\beta}{k}-1}\mathcal{F}(u)du, y > \mu_1, \tag{17}$$

and

$$\left({}^{(k,\psi)}\mathcal{J}^{\beta}_{\mu_2^-}\mathcal{F}\right)(y) = \frac{1}{k\Gamma_k(\beta)}\int_{y}^{\mu_2}\psi'(u)\Big[\psi(u)-\psi(y)\Big]^{\frac{\beta}{k}-1}\mathcal{F}(u)du, y < \mu_2, \tag{18}$$

where $\Gamma_k(\beta) = \int_0^\infty e^{\frac{-x^k}{k}}x^{\beta-1}dx$ is k-gamma function [18]. The k-gamma function satisfies the following properties:

1. $\Gamma_k(k) = 1$;
2. $\Gamma_k(\beta) = k^{\frac{\beta}{k}-1}\Gamma(\frac{\beta}{k})$;
3. $\Gamma_k(\beta+k) = \beta\Gamma_k(\beta)$;
4. $\Gamma_k(\beta) = \Gamma(\beta), k \to 1$.

The notations $\left({}^{(k,\psi)}\mathcal{J}^{\beta}_{\mu_1^+}\mathcal{F}\right)(y)$ and $\left({}^{(k,\psi)}\mathcal{J}^{\beta}_{\mu_2^-}\mathcal{F}\right)(y)$ are, respectively, called the left- and right-sided (k,ψ)- Riemann–Liouville fractional integrals of a function $\mathcal{F}$ of order β.

Remark 2. *If $k = 1$, Definition 7 reduces to Definition 3.*

Definition 8 ([19])**.** *For an integrable function $\mathcal{F}$ and for a strictly increasing continuous function ψ on $[\mu_1, \mu_2]$, let $\delta \in (0,1]$, we have for all $\beta \in \mathbb{C}, Re(\beta) \geq 0, k \in \mathbb{R}^+$*

$$\left({}^{(k,\psi)}\mathcal{J}^{\beta,\delta}_{\mu_1^+}\mathcal{F}\right)(y) = \frac{1}{\delta^{\frac{\beta}{k}}k\Gamma_k(\beta)}\int_{\mu_1}^{y} exp\left[\frac{\delta-1}{\delta}(\psi(y)-\psi(u))\right]\Big(\psi(y)-\psi(u)\Big)^{\frac{\beta}{k}-1}\psi'(u)\mathcal{F}(u)du, y > \mu_1, \tag{19}$$

and

$$\left({}^{(k,\psi)}\mathcal{J}^{\beta,\delta}_{\mu_2^-}\mathcal{F}\right)(y) = \frac{1}{\delta^{\frac{\beta}{k}}k\Gamma_k(\beta)}\int_{y}^{\mu_2} exp\left[\frac{\delta-1}{\delta}(\psi(u)-\psi(y))\right]\Big(\psi(u)-\psi(y)\Big)^{\frac{\beta}{k}-1}\psi'(u)\mathcal{F}(u)du, y < \mu_2. \tag{20}$$

The notations $\left({}^{(k,\psi)}\mathcal{J}^{\beta,\delta}_{\mu_1^+}\mathcal{F}\right)(y)$ and $\left({}^{(k,\psi)}\mathcal{J}^{\beta,\delta}_{\mu_2^-}\mathcal{F}\right)(y)$ are called, respectively, the left- and right-sided proportional k-fractional integral of $\mathcal{F}$ with respect to the function ψ of order β.

Definition 9 ([19]). *Let $\beta \in \mathbb{C}, Re(\beta) \geq 0, k \in \mathbb{R}^+, \delta > 0$ and $m \in \mathbb{N}$ such that $m = [Re(\frac{\beta}{k})] + 1$. Then, for any $f \in L^1[a,b]$ and $\psi \in C^1[a,b]$ where $\psi(t) > 0$ we have*

$$\begin{aligned}\left({}^{(k,\psi)}D^{\beta,\delta}_{a^+}f\right)(y) &= \left(\frac{\delta k}{\psi'(y)}\frac{d}{dy}\right)^m \left({}^{(k,\psi)}\mathcal{J}^{m-\beta,\delta}_{a^+}f\right)(y) \\ &= \frac{{}^{(k,\psi)}D^{m,\delta}_y}{\delta^{\frac{m-\beta}{k}}k\Gamma_k(m-\beta)}\int_a^y exp\left[\frac{\delta-1}{\delta}(\psi(y)-\psi(u))\right]\left(\psi(y)-\psi(u)\right)^{\frac{m-\beta}{k}-1}\psi'(u)f(u)du,\end{aligned} \tag{21}$$

and

$$\begin{aligned}\left({}^{(k,\psi)}D^{\beta,\delta}_{b^-}f\right)(y) &= \left(\frac{-\delta k}{\psi'(y)}\frac{d}{dy}\right)^m \left({}^{(k,\psi)}\mathcal{J}^{m-\beta,\delta}_{b^-}f\right)(y) \\ &= \frac{{}^{(k,\psi)}D^{m,\delta}_\Theta}{\delta^{\frac{m-\beta}{k}}k\Gamma_k(m-\beta)}\int_y^b exp\left[\frac{\delta-1}{\delta}(\psi(u)-\psi(y))\right]\left(\psi(u)-\psi(y)\right)^{\frac{m-\beta}{k}-1}\psi'(u)f(u)du,\end{aligned} \tag{22}$$

where

$$\left({}^{(k,\psi)}D^{m,\delta}_y\right) = \underbrace{{}^{(k,\psi)}D^{\delta}_y\,{}^{(k,\psi)}D^{\delta}_y \ldots {}^{(k,\psi)}D^{\delta}_y}_{m-times} \left({}^{(k,\psi)}D^{m,\delta}_\Theta\right) = \underbrace{{}^{(k,\psi)}D^{\delta}_\Theta\,{}^{(k,\psi)}D^{\delta}_\Theta \ldots {}^{(k,\psi)}D^{\delta}_\Theta}_{m-times} \tag{23}$$

The notations $\left({}^{(k,\psi)}D^{\beta,\delta}_{a^+}f\right)(y)$ and $\left({}^{(k,\psi)}D^{\beta,\delta}_{b^-}f\right)(y)$ are, respectively, called the left- and right-sided (k,ψ)-proportional fractional integral of a function f with respect to ψ of order β and type δ.

Lemma 2 ([19]). *Let $\alpha, \beta \in \mathbb{C}$ be such that $Re(\alpha) > 0$ and $Re(\beta) > 0$ and $k \in \mathbb{R}^+$. Then, for any $\delta \in (0,1]$, we have*

$$\left({}^{(k,\psi)}\mathcal{J}^{\alpha,\delta}_{\mu_1^+}\right)\left({}^{(k,\psi)}\mathcal{J}^{\beta,\delta}_{\mu_1^+}\right)\mathcal{F} = \left({}^{(k,\psi)}\mathcal{J}^{\alpha+\beta,\delta}_{\mu_1^+}\right)\mathcal{F}. \tag{24}$$

3. Main Results

Let $\delta \in (0,1], \beta \in \mathbb{C}, Re(\beta) \geq 0$ and ψ be a continuous strictly increasing function. Then, by using Definition 8, the left-sided (k,ψ)-proportional fractional integral for any constant C is given by

$$\left({}^{(k,\psi)}\mathcal{J}^{\beta,\delta}_{x^+}C\right)(y) = \frac{(\psi(y)-\psi(x))^{\frac{\beta}{k}}}{\delta^{\frac{\beta}{k}}\Gamma_k(\beta+k)}C. \tag{25}$$

The following Lemma will be very useful in obtaining our main results.

Lemma 3. *Let $\delta \in (0,1], \beta \in \mathbb{C}, Re(\beta) \geq 0, k > 0$, and ψ be a continuous strictly increasing function. Then, we have*

$$\int_0^1 exp\left[\frac{\delta-1}{\delta}\xi(y-x)\right]\xi^{\frac{\beta}{k}-1}d\xi = \frac{k}{\beta}.$$

Proof. Using Equation (25) and letting $\xi = \frac{y-\psi(u)}{y-x}$, we have $d\xi = -\frac{\psi'(u)}{y-x}du$. When $\xi = 0$ and $\xi = 1$ then $u = \psi^{-1}(y)$ and $\psi^{-1}(x)$, respectively. Thus,

$$\int_0^1 exp\Big[\frac{\delta-1}{\delta}\xi(y-x)\Big]\xi^{\frac{\beta}{k}-1}d\xi = \frac{1}{(y-x)^{\frac{\beta}{k}}}\int_{\psi^{-1}(x)}^{\psi^{-1}(y)} exp\Big[\frac{\delta-1}{\delta}(y-\psi(u))\Big]\Big(y-\psi(u)\Big)^{\frac{\beta}{k}-1}\psi'(u)du$$
$$= \frac{\delta^{\frac{\beta}{k}}k\Gamma_k(\beta)}{(y-x)^{\frac{\beta}{k}}}\frac{1}{\delta^{\frac{\beta}{k}}k\Gamma_k(\beta)}\int_{\psi^{-1}(x)}^{\psi^{-1}(y)} exp\Big[\frac{\delta-1}{\delta}(y-\psi(u))\Big]\Big(y-\psi(u)\Big)^{\frac{\beta}{k}-1}\psi'(u)du$$
$$= \frac{\delta^{\frac{\beta}{k}}k\Gamma_k(\beta)}{(y-x)^{\frac{\beta}{k}}}\frac{\Big(\psi(\psi^{-1}(y))-\psi(\psi^{-1}(x))\Big)^{\frac{\beta}{k}}}{\delta^{\frac{\beta}{k}}\Gamma_k(\beta+k)} = \frac{k}{\beta}.$$

□

Theorem 2. *Let $\mathcal{F}:[\mu_1,\mu_2]\to\mathbb{R}$ be a convex differentiable function on (μ_1,μ_2), and $\psi:[\mu_1,\mu_2]\to[\mu_1,\mu_2]$ be a continuous strictly increasing function with $0\le\mu_1<\mu_2$ and $(\mathcal{F}\circ\psi):[\mu_1,\mu_2]\to\mathbb{R}$ be an integrable function on $[\mu_1,\mu_2]$. Then, for all $x,y\in[\mu_1,\mu_2]$ the following inequalities hold:*

$$\begin{aligned}
&\mathcal{F}\Big(\mu_1+\mu_2-\frac{x+y}{2}\Big)\\
&\le \mathcal{F}(\mu_1)+\mathcal{F}(\mu_2)-\frac{\delta^{\frac{\beta}{k}}\Gamma_k(\beta+k)}{2(y-x)^{\frac{\beta}{k}}}\Big[{}^{(k,\psi)}\mathcal{J}^{\beta,\delta}_{(\psi^{-1}(x))^+}(\mathcal{F}\circ\psi)(\psi^{-1}(y))+{}^{(k,\psi)}\mathcal{J}^{\beta,\delta}_{(\psi^{-1}(y))^-}(\mathcal{F}\circ\psi)(\psi^{-1}(x))\Big]\\
&\le \mathcal{F}(\mu_1)+\mathcal{F}(\mu_2)-\mathcal{F}\Big(\frac{x+y}{2}\Big).
\end{aligned}$$

Proof. Using Jensen–Mercer's inequality on the function $\mathcal{F}$, for $\lambda_1,\lambda_2\in[\mu_1,\mu_2]$ we obtain

$$\mathcal{F}\Big(\mu_1+\mu_2-\frac{\lambda_1+\lambda_2}{2}\Big)\le\mathcal{F}(\mu_1)+\mathcal{F}(\mu_2)-\frac{\mathcal{F}(\lambda_1)+\mathcal{F}(\lambda_2)}{2}.$$

Now, letting $\lambda_1=\xi x+(1-\xi)y$ and $\lambda_2=(1-\xi)x+\xi y$ with $\xi\in[0,1]$, we obtain

$$\mathcal{F}\Big(\mu_1+\mu_2-\frac{x+y}{2}\Big)\le\mathcal{F}(\mu_1)+\mathcal{F}(\mu_2)-\frac{\mathcal{F}\Big(\xi x+(1-\xi)y\Big)+\mathcal{F}\Big((1-\xi)x+\xi y\Big)}{2}, \tag{26}$$

Multiplying both sides of the inequality (26) by $exp\Big[\frac{\delta-1}{\delta}\xi(y-x)\Big]\xi^{\frac{\beta}{k}-1}$ and integrating with respect to ξ over $[0,1]$, we obtain

$$\begin{aligned}
&\int_0^1 exp\Big[\frac{\delta-1}{\delta}\xi(y-x)\Big]\xi^{\frac{\beta}{k}-1}\mathcal{F}\Big(\mu_1+\mu_2-\frac{x+y}{2}\Big)d\xi\\
&\le\int_0^1 exp\Big[\frac{\delta-1}{\delta}\xi(y-x)\Big]\xi^{\frac{\beta}{k}-1}\Big(\mathcal{F}(\mu_1)+\mathcal{F}(\mu_2)\Big)d\xi\\
&-\int_0^1 exp\Big[\frac{\delta-1}{\delta}\xi(y-x)\Big]\xi^{\frac{\beta}{k}-1}\frac{\mathcal{F}\Big(\xi x+(1-\xi)y\Big)+\mathcal{F}\Big((1-\xi)x+\xi y\Big)}{2}d\xi.
\end{aligned}\tag{27}$$

Using Lemma 3 on (27) we obtain the following

$$\begin{aligned}
&\mathcal{F}\Big(\mu_1+\mu_2-\frac{x+y}{2}\Big)\\
&\le\mathcal{F}(\mu_1)+\mathcal{F}(\mu_2)-\frac{\beta}{2k}\Bigg[\int_0^1 exp\Big[\frac{\delta-1}{\delta}\xi(y-x)\Big]\xi^{\frac{\beta}{k}-1}\mathcal{F}(\xi x+(1-\xi)y)d\xi\\
&+\int_0^1 exp\Big[\frac{\delta-1}{\delta}\xi(y-x)\Big]\xi^{\frac{\beta}{k}-1}\mathcal{F}((1-\xi)x+\xi y)d\xi\Bigg].
\end{aligned}$$

Now,

$$\frac{\delta^{\frac{\beta}{k}}\Gamma_k(\beta+k)}{2(y-x)^{\frac{\beta}{k}}}\left[{}^{(k,\psi)}\mathcal{J}^{\beta,\delta}_{(\psi^{-1}(x))^+}(\mathcal{F}\circ\psi)(\psi^{-1}(y))+{}^{(k,\psi)}\mathcal{J}^{\beta,\delta}_{(\psi^{-1}(y))^-}(\mathcal{F}\circ\psi)(\psi^{-1}(x))\right]$$
$$=\frac{\delta^{\frac{\beta}{k}}\Gamma_k(\beta+k)}{2(y-x)^{\frac{\beta}{k}}}\frac{1}{\delta^{\frac{\beta}{k}}k\Gamma_k(\beta)}\left[\int_{\psi^{-1}(x)}^{\psi^{-1}(y)}exp\Big[\frac{\delta-1}{\delta}(y-\psi(u))\Big]\Big(y-\psi(u)\Big)^{\frac{\beta}{k}-1}(\mathcal{F}\circ\psi)(u)\psi'(u)du\right.$$
$$\left.+\int_{\psi^{-1}(x)}^{\psi^{-1}(y)}exp\Big[\frac{\delta-1}{\delta}(\psi(v)-x)\Big]\Big(\psi(v)-x\Big)^{\frac{\beta}{k}-1}(\mathcal{F}\circ\psi)(v)\psi'(v)dv\right]$$
$$=\frac{\beta}{2k}\left[\int_{\psi^{-1}(x)}^{\psi^{-1}(y)}exp\Big[\frac{\delta-1}{\delta}(y-\psi(u))\Big]\Big(\frac{y-\psi(u)}{y-x}\Big)^{\frac{\beta}{k}-1}(\mathcal{F}\circ\psi)(u)\frac{\psi'(u)}{y-x}du\right.$$
$$\left.+\int_{\psi^{-1}(x)}^{\psi^{-1}(y)}exp\Big[\frac{\delta-1}{\delta}(\psi(v)-x)\Big]\Big(\frac{\psi(v)-x}{y-x}\Big)^{\frac{\beta}{k}-1}(\mathcal{F}\circ\psi)(v)\frac{\psi'(v)}{y-x}dv\right].$$

Putting

$$\psi(u)=\xi x+(1-\xi)y\text{and}\psi(v)=(1-\xi)x+\xi y \tag{28}$$

we obtain

$$\mathcal{F}(\mu_1)+\mathcal{F}(\mu_2)-\frac{\delta^{\frac{\beta}{k}}\Gamma_k(\beta+k)}{2(y-x)^{\frac{\beta}{k}}}\left[{}^{(k,\psi)}\mathcal{J}^{\beta,\delta}_{(\psi^{-1}(x))^+}(\mathcal{F}\circ\psi)(\psi^{-1}(y))+{}^{(k,\psi)}\mathcal{J}^{\beta,\delta}_{(\psi^{-1}(y))^-}(\mathcal{F}\circ\psi)(\psi^{-1}(x))\right]$$
$$=\mathcal{F}(\mu_1)+\mathcal{F}(\mu_2)-\frac{\beta}{2k}\left[\int_0^1 exp\Big[\frac{\delta-1}{\delta}\xi(y-x)\Big]\xi^{\frac{\beta}{k}-1}\mathcal{F}\Big(\xi x+(1-\xi)y\Big)d\xi\right.$$
$$\left.+\int_0^1 exp\Big[\frac{\delta-1}{\delta}\xi(y-x)\Big]\xi^{\frac{\beta}{k}-1}\mathcal{F}\Big((1-\xi)x+\xi y\Big)d\xi\right]$$
$$\geq\mathcal{F}\Big(\mu_1+\mu_2-\frac{x+y}{2}\Big),$$

which complete the proof of the first inequality.

To prove the second inequality, by using convexity of $\mathcal{F}$, we can write

$$\mathcal{F}\Big(\frac{\psi(u)+\psi(v)}{2}\Big)\leq\frac{(\mathcal{F}\circ\psi)(u)+(\mathcal{F}\circ\psi)(v)}{2},$$

then taking $\psi(u)$ and $\psi(v)$ given in (28) we obtain

$$\mathcal{F}\Big(\frac{x+y}{2}\Big)\leq\frac{\mathcal{F}\Big(\xi x+(1-\xi)y\Big)+\mathcal{F}\Big((1-\xi)x+\xi y\Big)}{2}. \tag{29}$$

Multiplying both sides of the inequality (29) by $exp\Big[\frac{\delta-1}{\delta}\xi(y-x)\Big]\xi^{\frac{\beta}{k}-1}$ and integrating with respect to ξ over $[0,1]$, we obtain

$$\mathcal{F}\Big(\frac{x+y}{2}\Big)$$
$$\leq\frac{\beta}{2k}\left[\int_0^1 exp\Big[\frac{\delta-1}{\delta}\xi(y-x)\Big]\xi^{\frac{\beta}{k}-1}\mathcal{F}\Big(\xi x+(1-\xi)y\Big)d\xi+\int_0^1 exp\Big[\frac{\delta-1}{\delta}\xi(y-x)\Big]\xi^{\frac{\beta}{k}-1}\mathcal{F}\Big((1-\xi)x+\eta y\Big)d\xi\right]$$
$$=\frac{\delta^{\frac{\beta}{k}}\Gamma_k(\beta+k)}{2(y-x)^{\frac{\beta}{k}}}\left[{}^{(k,\psi)}\mathcal{J}^{\beta,\delta}_{(\psi^{-1}(x))^+}(\mathcal{F}\circ\psi)(\psi^{-1}(y))+{}^{(k,\psi)}\mathcal{J}^{\beta,\delta}_{(\psi^{-1}(y))^-}(\mathcal{F}\circ\psi)(\psi^{-1}(x))\right]. \tag{30}$$

Multiplying both sides of inequality (30) by -1 and adding $\mathcal{F}(\mu_1)+\mathcal{F}(\mu_2)$, we obtain

$$\begin{aligned}
&\mathcal{F}(\mu_1)+\mathcal{F}(\mu_2)-\mathcal{F}\Big(\frac{x+y}{2}\Big)\\
&\geq \mathcal{F}(\mu_1)+\mathcal{F}(\mu_2)-\frac{\delta^{\frac{\beta}{k}}\Gamma_k(\beta+k)}{2(y-x)^{\frac{\beta}{k}}}\Big[{}^{(k,\psi)}\mathcal{J}^{\beta,\delta}_{(\psi^{-1}(x))^+}(\mathcal{F}\circ\psi)(\psi^{-1}(y))+{}^{(k,\psi)}\mathcal{J}^{\beta,\delta}_{(\psi^{-1}(y))^-}(\mathcal{F}\circ\psi)(\psi^{-1}(x))\Big],
\end{aligned}$$

which proves the second inequality. □

Remark 3. *In Theorem 2 by putting*

a. $\delta=1$ *we obtain its ψ-Riemann–Liouville k-fractional integral version,*

$$\begin{aligned}
&\mathcal{F}\Big(\mu_1+\mu_2-\frac{x+y}{2}\Big)\\
&\leq \mathcal{F}(\mu_1)+\mathcal{F}(\mu_2)-\frac{\Gamma_k(\beta+k)}{2(y-x)^{\frac{\beta}{k}}}\Big[{}^{(k,\psi)}\mathcal{J}^{\beta}_{(\psi^{-1}(x))^+}(\mathcal{F}\circ\psi)(\psi^{-1}(y))+{}^{(k,\psi)}\mathcal{J}^{\beta}_{(\psi^{-1}(y))^-}(\mathcal{F}\circ\psi)(\psi^{-1}(x))\Big]\\
&\leq \mathcal{F}(\mu_1)+\mathcal{F}(\mu_2)-\mathcal{F}\Big(\frac{x+y}{2}\Big),
\end{aligned}$$

which is proved in [10].

b. $\delta=k=1$ *and* $\psi(x)=x$ *we obtain,*

$$\begin{aligned}
\mathcal{F}\Big(\mu_1+\mu_2-\frac{x+y}{2}\Big)&\leq \mathcal{F}(\mu_1)+\mathcal{F}(\mu_2)-\frac{\Gamma(\beta+1)}{2(y-x)^{\beta}}\Big[\mathcal{J}^{\beta}_{x^+}\mathcal{F}(y)+\mathcal{J}^{\beta}_{y^-}\mathcal{F}(x)\Big]\\
&\leq \mathcal{F}(\mu_1)+\mathcal{F}(\mu_2)-\mathcal{F}\Big(\frac{x+y}{2}\Big),
\end{aligned}$$

which is proved in [20].

c. $\delta=\beta=k=1$ *and* $\psi(x)=x$ *we obtain,*

$$\begin{aligned}
\mathcal{F}\Big(\mu_1+\mu_2-\frac{x+y}{2}\Big)&\leq \mathcal{F}(\mu_1)+\mathcal{F}(\mu_2)-\int_0^1\mathcal{F}\Big(\xi x+(1-\xi)y\Big)d\xi\\
&\leq \mathcal{F}(\mu_1)+\mathcal{F}(\mu_2)-\mathcal{F}\Big(\frac{x+y}{2}\Big),
\end{aligned}$$

which is proved in [21].

Theorem 3. *Suppose that the hypothesis of Theorem 2 hold. Then,*

$$\begin{aligned}
&\mathcal{F}\Big(\mu_1+\mu_2-\frac{x+y}{2}\Big)\\
&\leq \frac{\delta^{\frac{\beta}{k}}\Gamma_k(\beta+k)}{2(y-x)^{\frac{\beta}{k}}}\Big[{}^{(k,\psi)}\mathcal{J}^{\beta,\delta}_{(\psi^{-1}(\mu_1+\mu_2-y))^+}(\mathcal{F}\circ\psi)(\psi^{-1}(\mu_1+\mu_2-x))\\
&+{}^{(k,\psi)}\mathcal{J}^{\beta,\delta}_{(\psi^{-1}(\mu_1+\mu_2-x))^-}(\mathcal{F}\circ\psi)(\psi^{-1}(\mu_1+\mu_2-y))\Big]\\
&\leq \frac{\mathcal{F}(\mu_1+\mu_2-y)+\mathcal{F}(\mu_1+\mu_2-x)}{2}\leq \mathcal{F}(\mu_1)+\mathcal{F}(\mu_2)-\frac{1}{2}\Big(\mathcal{F}(x)+\mathcal{F}(y)\Big).
\end{aligned}$$

Proof. From the convexity of the function $\mathcal{F}$, for all $\lambda_1,\lambda_2\in[\mu_1,\mu_2]$, we have

$$\mathcal{F}\Big(\mu_1+\mu_2-\frac{\lambda_1+\lambda_2}{2}\Big)=\mathcal{F}\Big(\frac{(\mu_1+\mu_2-\lambda_1)+(\mu_1+\mu_2-\lambda_2)}{2}\Big)\leq\frac{1}{2}\Big[\mathcal{F}(\mu_1+\mu_2-\lambda_1)+\mathcal{F}(\mu_1+\mu_2-\lambda_2)\Big].$$

Let $\mu_1+\mu_2-\lambda_1 = \xi(\mu_1+\mu_2-x)+(1-\xi)(\mu_1+\mu_2-y)$ and $\mu_1+\mu_2-\lambda_2 = (1-\xi)(\mu_1+\mu_2-x)+\xi(\mu_1+\mu_2-y)$ which is $\lambda_1 = y+\xi(x-y)$ and $\lambda_2 = x+\xi(y-x)$. Then,

$$\mathcal{F}\Big(\mu_1+\mu_2-\frac{x+y}{2}\Big) \leq \frac{1}{2}\Big[\mathcal{F}\Big(\xi(\mu_1+\mu_2-x)+(1-\xi)(\mu_1+\mu_2-y)\Big)+\mathcal{F}\Big((1-\xi)(\mu_1+\mu_2-x)+\xi(\mu_1+\mu_2-y)\Big)\Big]. \tag{31}$$

Multiplying both sides of the inequality (31) by $exp\Big[\frac{\delta-1}{\delta}\xi(y-x)\Big]\xi^{\frac{\beta}{k}-1}$ and integrating with respect to ξ over $[0,1]$, we obtain

$$\begin{aligned}
&\mathcal{F}\left(\mu_1+\mu_2-\frac{x+y}{2}\right)\\
&\leq \frac{\beta}{2k}\left[\int_0^1 exp\Big[\frac{\delta-1}{\delta}\xi(y-x)\Big]\xi^{\frac{\beta}{k}-1}\mathcal{F}\Big(\xi(\mu_1+\mu_2-x)+(1-\xi)(\mu_1+\mu_2-y)\Big)d\xi\right.\\
&\left.+\int_0^1 exp\Big[\frac{\delta-1}{\delta}\xi(y-x)\Big]\xi^{\frac{\beta}{k}-1}\mathcal{F}\Big((1-\xi)(\mu_1+\mu_2-x)+\xi(\mu_1+\mu_2-y)\Big)d\eta\right].
\end{aligned}$$

Now,

$$\begin{aligned}
&\frac{\delta^{\frac{\beta}{k}}\Gamma_k(\beta+k)}{2(y-x)^{\frac{\beta}{k}}}\Big[{}^{(k,\psi)}\mathcal{J}^{\beta,\delta}_{(\psi^{-1}(\mu_1+\mu_2-y))^+}(\mathcal{F}\circ\psi)(\psi^{-1}(\mu_1+\mu_2-x))+{}^{(k,\psi)}\mathcal{J}^{\beta,\delta}_{(\psi^{-1}(\mu_1+\mu_2-x))^-}(\mathcal{F}\circ\psi)(\psi^{-1}(\mu_1+\mu_2-y))\Big]\\
&=\frac{\delta^{\frac{\beta}{k}}\Gamma_k(\beta+k)}{2(y-x)^{\frac{\beta}{k}}\delta^{\frac{\beta}{k}}k\Gamma_k(\beta)}\left[\int_{\psi^{-1}(\mu_1+\mu_2-y)}^{\psi^{-1}(\mu_1+\mu_2-x)} exp\Big[\frac{\delta-1}{\delta}((\mu_1+\mu_2-x)-\psi(u))\Big]\Big((\mu_1+\mu_2-x)-\psi(u)\Big)^{\frac{\beta}{k}-1}(\mathcal{F}\circ\psi)(u)\psi'(u)du\right.\\
&\left.+\int_{\psi^{-1}(\mu_1+\mu_2-y)}^{\psi^{-1}(\mu_1+\mu_2-x)} exp\Big[\frac{\delta-1}{\delta}(\psi(v)-(\mu_1+\mu_2-y))\Big]\Big(\psi(v)-(\mu_1+\mu_2-y)\Big)^{\frac{\beta}{k}-1}(\mathcal{F}\circ\psi)(v)\psi'(v)dv\right]\\
&=\frac{\beta}{2k}\left[\int_{\psi^{-1}(\mu_1+\mu_2-y)}^{\psi^{-1}(\mu_1+\mu_2-x)} exp\Big[\frac{\delta-1}{\delta}((\mu_1+\mu_2-x)-\psi(u))\Big]\Big(\frac{(\mu_1+\mu_2-x)-\psi(u)}{y-x}\Big)^{\frac{\beta}{k}-1}(\mathcal{F}\circ\psi)(u)\frac{\psi'(u)}{y-x}du\right.\\
&\left.+\int_{\psi^{-1}(\mu_1+\mu_2-y)}^{\psi^{-1}(\mu_1+\mu_2-x)} exp\Big[\frac{\delta-1}{\delta}(\psi(v)-(\mu_1+\mu_2-y))\Big]\Big(\frac{\psi(v)-(\mu_1+\mu_2-y)}{y-x}\Big)^{\frac{\beta}{k}-1}(\mathcal{F}\circ\psi)(v)\frac{\psi'(v)}{y-x}dv\right].
\end{aligned}$$

Putting $\psi(u) = \xi(\mu_1+\mu_2-y)+(1-\xi)(\mu_1+\mu_2-x)$ and $\psi(v) = (1-\xi)(\mu_1+\mu_2-y)+\xi(\mu_1+\mu_2-x)$, we obtain

$$\begin{aligned}
&\frac{\delta^{\frac{\beta}{k}}\Gamma_k(\beta+k)}{2(y-x)^{\frac{\beta}{k}}}\Big[{}^{(k,\psi)}\mathcal{J}^{\beta,\delta}_{(\psi^{-1}(\mu_1+\mu_2-y))^+}(\mathcal{F}\circ\psi)(\psi^{-1}(\mu_1+\mu_2-x))+{}^{(k,\psi)}\mathcal{J}^{\beta,\delta}_{(\psi^{-1}(\mu_1+\mu_2-x))^-}(\mathcal{F}\circ\psi)(\psi^{-1}(\mu_1+\mu_2-y))\Big]\\
&=\frac{\beta}{2k}\left[\int_0^1 exp\Big[\frac{\delta-1}{\delta}\xi(y-x)\Big]\xi^{\frac{\beta}{k}-1}\mathcal{F}\Big(\xi(\mu_1+\mu_2-x)+(1-\xi)(\mu_1+\mu_2-y)\Big)d\xi\right.\\
&\left.+\int_0^1 exp\Big[\frac{\delta-1}{\delta}\xi(y-x)\Big]\xi^{\frac{\beta}{k}-1}\mathcal{F}\Big((1-\xi)(\mu_1+\mu_2-x)+\xi(\mu_1+\mu_2-y)\Big)d\xi\right]\\
&\geq\mathcal{F}\Big(\mu_1+\mu_2-\frac{x+y}{2}\Big),
\end{aligned}\tag{32}$$

which completes the proof of the first inequality.
To prove the second inequality we use the convexity of the function $\mathcal{F}$ as follows

$$\mathcal{F}\Big(\xi(\mu_1+\mu_2-x)+(1-\xi)(\mu_1+\mu_2-y)\Big)\leq\xi\mathcal{F}(\mu_1+\mu_2-x)+(1-\xi)\mathcal{F}(\mu_1+\mu_2-y). \tag{33}$$

and

$$\mathcal{F}\Big(\xi(\mu_1+\mu_2-y)+(1-\xi)(\mu_1+\mu_2-x)\Big)\leq\xi\mathcal{F}(\mu_1+\mu_2-y)+(1-\xi)\mathcal{F}(\mu_1+\mu_2-x). \tag{34}$$

Adding (33) and (34), we obtain

$$\mathcal{F}\Big(\xi(\mu_1+\mu_2-x)+(1-\xi)(\mu_1+\mu_2-y)\Big)+\mathcal{F}\Big(\xi(\mu_1+\mu_2-y)+(1-\xi)(\mu_1+\mu_2-x)\Big)$$
$$\leq \mathcal{F}(\mu_1+\mu_2-x)+\mathcal{F}(\mu_1+\mu_2-y)\leq 2\Big[\mathcal{F}(\mu_1)+\mathcal{F}(\mu_2)\Big]-\Big[\mathcal{F}(x)+\mathcal{F}(y)\Big]. \quad (35)$$

Multiplying both sides of the inequality (35) by $exp\Big[\frac{\delta-1}{\delta}\xi(y-x)\Big]\xi^{\frac{\beta}{k}-1}$ and integrating with respect to ξ over $[0,1]$, we obtain

$$\frac{\beta}{2k}\Bigg[\int_0^1 exp\Big[\frac{\delta-1}{\delta}\xi(y-x)\Big]\xi^{\frac{\beta}{k}-1}\mathcal{F}\Big(\xi(\mu_1+\mu_2-x)+(1-\xi)(\mu_1+\mu_2-y)\Big)d\xi$$
$$+\int_0^1 exp\Big[\frac{\delta-1}{\delta}\xi(y-x)\Big]\xi^{\frac{\beta}{k}-1}\mathcal{F}\Big((1-\xi)(\mu_1+\mu_2-x)+\xi(\mu_1+\mu_2-y)\Big)d\xi\Bigg]$$
$$\leq \frac{\mathcal{F}(\mu_1+\mu_2-x)+\mathcal{F}(\mu_1+\mu_2-y)}{2}\leq \Big[\mathcal{F}(\mu_1)+\mathcal{F}(\mu_2)\Big]-\frac{1}{2}\Big[\mathcal{F}(x)+\mathcal{F}(y)\Big].$$

Thus, using Equation (32) we obtain that

$$\frac{\delta^{\frac{\beta}{k}}\Gamma_k(\beta+k)}{2(y-x)^{\frac{\beta}{k}}}\Big[{}^{(k,\psi)}\mathcal{J}^{\beta,\delta}_{(\psi^{-1}(\mu_1+\mu_2-y))^+}(\mathcal{F}\circ\psi)(\psi^{-1}(\mu_1+\mu_2-x))+{}^{(k,\psi)}\mathcal{J}^{\beta,\delta}_{(\psi^{-1}(\mu_1+\mu_2-x))^-}(\mathcal{F}\circ\psi)(\psi^{-1}(\mu_1+\mu_2-y))\Big]$$
$$=\frac{\beta}{2k}\Bigg[\int_0^1 exp\Big[\frac{\delta-1}{\delta}\xi(y-x)\Big]\xi^{\frac{\beta}{k}-1}\mathcal{F}\Big(\xi(\mu_1+\mu_2-x)+(1-\xi)(\mu_1+\mu_2-y)\Big)d\xi$$
$$+\int_0^1 exp\Big[\frac{\delta-1}{\delta}\xi(y-x)\Big]\xi^{\frac{\beta}{k}-1}\mathcal{F}\Big((1-\xi)(\mu_1+\mu_2-x)+\xi(\mu_1+\mu_2-y)\Big)d\xi\Bigg]$$
$$\leq \frac{\mathcal{F}(\mu_1+\mu_2-x)+\mathcal{F}(\mu_1+\mu_2-y)}{2}\leq \Big[\mathcal{F}(\mu_1)+\mathcal{F}(\mu_2)\Big]-\frac{1}{2}\Big[\mathcal{F}(x)+\mathcal{F}(y)\Big],$$

which completes the proof of the second inequality. □

Remark 4. *In Theorem 3 by putting*

a. *$\delta=1$ we obtain its ψ-Riemann–Liouville k-fractional integral version,*

$$\mathcal{F}\Big(\mu_1+\mu_2-\frac{x+y}{2}\Big)$$
$$\leq\frac{\Gamma_k(\beta+k)}{2(y-x)^{\frac{\beta}{k}}}\Big[{}^{(k,\psi)}\mathcal{J}^{\beta}_{(\psi^{-1}(\mu_1+\mu_2-y))^+}(\mathcal{F}\circ\psi)(\psi^{-1}(\mu_1+\mu_2-x))$$
$$+{}^{(k,\psi)}\mathcal{J}^{\beta}_{(\psi^{-1}(\mu_1+\mu_2-x))^-}(\mathcal{F}\circ\psi)(\psi^{-1}(\mu_1+\mu_2-y))\Big]$$
$$\leq\frac{\mathcal{F}(\mu_1+\mu_2-y)+\mathcal{F}(\mu_1+\mu_2-x)}{2}\leq\mathcal{F}(\mu_1)+\mathcal{F}(\mu_2)-\frac{\mathcal{F}(x)+\mathcal{F}(y)}{2},$$

which is proved in [10].

b. *$\delta=k=1$ and $\psi(x)=x$ we obtain,*

$$\mathcal{F}\Big(\mu_1+\mu_2-\frac{x+y}{2}\Big)\leq\frac{\Gamma(\beta+1)}{2(y-x)^{\beta}}\Big[\mathcal{J}^{\beta}_{(\mu_1+\mu_2-y)^+}\mathcal{F}(\mu_1+\mu_2-x)+\mathcal{J}^{\beta}_{(\mu_1+\mu_2-x)^-}\mathcal{F}(\mu_1+\mu_2-y)\Big]$$
$$\leq\frac{\mathcal{F}(\mu_1+\mu_2-y)+\mathcal{F}(\mu_1+\mu_2-x)}{2}\leq\mathcal{F}(\mu_1)+\mathcal{F}(\mu_2)-\frac{\mathcal{F}(x)+\mathcal{F}(y)}{2},$$

which is proved in [20].

c. $\delta = \beta = k = 1$ *and* $\psi(x) = x$ *we obtain,*

$$\begin{aligned}\mathcal{F}\Big(\mu_1 + \mu_2 - \frac{x+y}{2}\Big) &\leq \frac{1}{y-x}\int_x^y \mathcal{F}(\mu_1 + \mu_2 - \xi)d\xi \\ &\leq \mathcal{F}(\mu_1) + \mathcal{F}(\mu_2) - \frac{\mathcal{F}(x) + \mathcal{F}(y)}{2},\end{aligned}$$

which is proved in [21].

Theorem 4. *Let* $\mathcal{F} : [\mu_1, \mu_2] \to \mathbb{R}$ *be a convex differentiable function on* (μ_1, μ_2)*,* $\psi : [\mu_1, \mu_2] \to [\mu_1, \mu_2] \subseteq \mathbb{R}$ *be a continuous strictly increasing function with* $0 \leq \mu_1 < \mu_2$ *and* $(\mathcal{F} \circ \psi) : [\mu_1, \mu_2] \to \mathbb{R}$ *be an integrable function on* $[\mu_1, \mu_2]$*. Then, for all* $x, y \in [\mu_1, \mu_2]$ *the following inequalities hold:*

$$\begin{aligned}&\mathcal{F}\left(\mu_1 + \mu_2 - \frac{x+y}{2}\right) \\ &\leq \frac{\delta^{\frac{\beta}{k}}\Gamma_k(\beta + k)}{2(y-x)^{\frac{\beta}{k}}}\left[{}^{(k,\psi)}\mathcal{J}^{\beta,\delta}_{(\psi^{-1}(\mu_1+\mu_2-\frac{x+y}{2}))^+}(\mathcal{F}\circ\psi)(\psi^{-1}(\mu_1 + \mu_2 - x))\right. \\ &\left.+ {}^{(k,\psi)}\mathcal{J}^{\beta,\delta}_{(\psi^{-1}(\mu_1+\mu_2-\frac{x+y}{2}))^-}(\mathcal{F}\circ\psi)(\psi^{-1}(\mu_1 + \mu_2 - y))\right] \\ &\leq \mathcal{F}(\mu_1) + \mathcal{F}(\mu_2) - \frac{\mathcal{F}(x) + \mathcal{F}(y)}{2}.\end{aligned}$$

Proof. By using convexity of the function $\mathcal{F}$ for all $\lambda_1, \lambda_2 \in [\mu_1, \mu_2]$, we have

$$\mathcal{F}\left(\mu_1 + \mu_2 - \frac{\lambda_1 + \lambda_2}{2}\right) = \mathcal{F}\left(\frac{(\mu_1 + \mu_2 - \lambda_1) + (\mu_1 + \mu_2 - \lambda_2)}{2}\right) \leq \frac{\mathcal{F}(\mu_1 + \mu_2 - \lambda_1) + \mathcal{F}(\mu_1 + \mu_2 - \lambda_2)}{2}. \quad (36)$$

Putting $\lambda_1 = \frac{\xi}{2}x + (1 - \frac{\xi}{2})y$ and $\lambda_2 = (1 - \frac{\xi}{2})x + \frac{\xi}{2}y$ in (36), we obtain

$$\mathcal{F}\left(\mu_1 + \mu_2 - \frac{x+y}{2}\right) \leq \frac{\mathcal{F}\Big(\mu_1 + \mu_2 - (\frac{\xi}{2}x + (1 - \frac{\xi}{2})y)\Big) + \mathcal{F}\Big(\mu_1 + \mu_2 - ((1 - \frac{\xi}{2})x + \frac{\xi}{2}y)\Big)}{2}. \quad (37)$$

Multiplying both sides of the inequality (37) by $exp\Big[\frac{\delta-1}{\delta}\frac{\xi}{2}(y-x)\Big](\frac{\xi}{2})^{\frac{\beta}{k}-1}$ and integrating with respect to ξ over $[0, 1]$, we obtain

$$\begin{aligned}&\mathcal{F}\Big(\mu_1 + \mu_2 - \frac{x+y}{2}\Big) \\ &\leq \frac{\beta}{2k}\left[\int_0^1 exp\Big[\frac{\delta-1}{\delta}\frac{\xi}{2}(y-x)\Big](\frac{\xi}{2})^{\frac{\beta}{k}-1}\mathcal{F}\Big(\mu_1 + \mu_2 - (\frac{\xi}{2}x + (1 - \frac{\xi}{2})y\Big)d\xi\right. \\ &\left.+ \int_0^1 exp\Big[\frac{\delta-1}{\delta}\frac{\xi}{2}(y-x)\Big](\frac{\xi}{2})^{\frac{\beta}{k}-1}\mathcal{F}\Big(\mu_1 + \mu_2 - ((1 - \frac{\xi}{2})x + \frac{\xi}{2}y\Big)d\xi\right].\end{aligned}$$

Now,

$$\frac{\delta^{\frac{\beta}{k}}\Gamma_k(\beta+k)}{2(y-x)^{\frac{\beta}{k}}}\Big[{}^{(k,\psi)}\mathcal{J}^{\beta,\delta}_{(\psi^{-1}(\mu_1+\mu_2-\frac{x+y}{2}))^+}(\mathcal{F}\circ\psi)(\psi^{-1}(\mu_1+\mu_2-x))$$
$$+{}^{(k,\psi)}\mathcal{J}^{\beta,\delta}_{(\psi^{-1}(\mu_1+\mu_2-\frac{x+y}{2}))^-}(\mathcal{F}\circ\psi)(\psi^{-1}(\mu_1+\mu_2-y))\Big]$$
$$=\frac{\delta^{\frac{\beta}{k}}\Gamma_k(\beta+k)}{2(y-x)^{\frac{\beta}{k}}}\frac{1}{\delta^{\frac{\beta}{k}}k\Gamma_k(\beta)}\Big[\int_{\psi^{-1}(\mu_1+\mu_2-\frac{x+y}{2})}^{\psi^{-1}(\mu_1+\mu_2-x)}exp\big[\frac{\delta-1}{\delta}((\mu_1+\mu_2-x)-\psi(u))\big]$$
$$\times\Big((\mu_1+\mu_2-x)-\psi(u)\Big)^{\frac{\beta}{k}-1}(\mathcal{F}\circ\psi)(u)\psi'(u)du$$
$$+\int_{\psi^{-1}(\mu_1+\mu_2-y)}^{\psi^{-1}(\mu_1+\mu_2-\frac{x+y}{2})}exp\big[\frac{\delta-1}{\delta}(\psi(v)-(\mu_1+\mu_2-y))\big]\Big(\psi(v)-(\mu_1+\mu_2-y)\Big)^{\frac{\beta}{k}-1}(\mathcal{F}\circ\psi)(v)\psi'(v)dv\Big]$$
$$=\frac{\beta}{2k}\Big[\int_{\psi^{-1}(\mu_1+\mu_2-\frac{x+y}{2})}^{\psi^{-1}(\mu_1+\mu_2-x)}exp\big[\frac{\delta-1}{\delta}((\mu_1+\mu_2-x)-\psi(u))\big]\Big(\frac{(\mu_1+\mu_2-x)-\psi(u)}{y-x}\Big)^{\frac{\beta}{k}-1}(\mathcal{F}\circ\psi)(u)\frac{\psi'(u)}{y-x}du$$
$$+\int_{\psi^{-1}(\mu_1+\mu_2-y)}^{\psi^{-1}(\mu_1+\mu_2-\frac{x+y}{2})}exp\big[\frac{\delta-1}{\delta}(\psi(v)-(\mu_1+\mu_2-y))\big]\Big(\frac{\psi(v)-(\mu_1+\mu_2-y)}{y-x}\Big)^{\frac{\beta}{k}-1}(\mathcal{F}\circ\psi)(v)\frac{\psi'(v)}{y-x}dv\Big].$$

Putting $\psi(u)=\mu_1+\mu_2-\big((1-\frac{\xi}{2})x+\frac{\xi}{2}y\big)$ and $\psi(v)=\mu_1+\mu_2-\big(\frac{\xi}{2}x+(1-\frac{\xi}{2})y\big)$, we obtain

$$\frac{\delta^{\frac{\beta}{k}}\Gamma_k(\beta+k)}{2(y-x)^{\frac{\beta}{k}}}\Big[{}^{(k,\psi)}\mathcal{J}^{\beta,\delta}_{(\psi^{-1}(\mu_1+\mu_2-\frac{x+y}{2}))^+}(\mathcal{F}\circ\psi)(\psi^{-1}(\mu_1+\mu_2-x))$$
$$+{}^{(k,\psi)}\mathcal{J}^{\beta,\delta}_{(\psi^{-1}(\mu_1+\mu_2-\frac{x+y}{2}))^-}(\mathcal{F}\circ\psi)(\psi^{-1}(\mu_1+\mu_2-y))\Big] \tag{38}$$
$$=\frac{\beta}{2k}\Big[\int_0^1 exp\big[\frac{\delta-1}{\delta}\frac{\xi}{2}(y-x)\big](\frac{\xi}{2})^{\frac{\beta}{k}-1}\mathcal{F}\Big(\mu_1+\mu_2-(\frac{\xi}{2}x+(1-\frac{\xi}{2})y)\Big)d\xi$$
$$+\int_0^1 exp\big[\frac{\delta-1}{\delta}\frac{\xi}{2}(y-x)\big](\frac{\xi}{2})^{\frac{\beta}{k}-1}\mathcal{F}\Big(\mu_1+\mu_2-((1-\frac{\xi}{2})x+\frac{\xi}{2}y)\Big)d\xi\Big] \tag{39}$$
$$\geq\mathcal{F}\Big(\mu_1+\mu_2-\frac{x+y}{2}\Big),$$

which proves the first inequality.
To prove the second inequality by using Jensen–Mercer inequality, we have

$$\mathcal{F}\Big(\mu_1+\mu_2-\Big(\frac{\xi}{2}x+\Big(1-\frac{\xi}{2}\Big)y\Big)\Big)\leq\mathcal{F}(\mu_1)+\mathcal{F}(\mu_2)-\Big(\frac{\xi}{2}\mathcal{F}(x)+\Big(1-\frac{\xi}{2}\Big)\mathcal{F}(y)\Big). \tag{40}$$

and

$$\mathcal{F}\Big(\mu_1+\mu_2-\Big(\Big(1-\frac{\xi}{2}\Big)x+\frac{\xi}{2}y\Big)\Big)\leq\mathcal{F}(\mu_1)+\mathcal{F}(\mu_2)-\Big(\Big(1-\frac{\xi}{2}\Big)\mathcal{F}(x)+\frac{\xi}{2}\mathcal{F}(y)\Big). \tag{41}$$

Adding the two inequalities (40) and (41), we obtain

$$\mathcal{F}\Big(\mu_1+\mu_2-\Big(\frac{\xi}{2}x+\Big(1-\frac{\xi}{2}\Big)y\Big)+\mathcal{F}\Big(\mu_1+\mu_2-\Big(\Big(1-\frac{\xi}{2}\Big)x+\frac{\xi}{2}y\Big)\Big)\leq 2\Big(\mathcal{F}(\mu_1)+\mathcal{F}(\mu_2)\Big)-\Big(\mathcal{F}(x)+\mathcal{F}(y)\Big). \tag{42}$$

Multiplying both sides of the inequality (42) by $exp\left[\frac{\delta-1}{\delta}\frac{\xi}{2}(y-x)\right]\left(\frac{\xi}{2}\right)^{\frac{\beta}{k}-1}$ and integrating with respect to ξ over $[0,1]$, we obtain

$$\frac{\beta}{2k}\left[\int_0^1 exp\left[\frac{\delta-1}{\delta}\frac{\xi}{2}(y-x)\right]\left(\frac{\xi}{2}\right)^{\frac{\beta}{k}-1}\mathcal{F}\left(\mu_1+\mu_2-\left(\frac{\xi}{2}x+\left(1-\frac{\xi}{2}\right)y\right)\right)d\xi\right.$$
$$\left.+\int_0^1 exp\left[\frac{\delta-1}{\delta}\frac{\xi}{2}(y-x)\right]\left(\frac{\xi}{2}\right)^{\frac{\beta}{k}-1}\mathcal{F}\left(\mu_1+\mu_2-\left(\left(1-\frac{\xi}{2}\right)x+\frac{\xi}{2}y\right)\right)d\xi\right]$$
$$\leq \mathcal{F}(\mu_1)+\mathcal{F}(\mu_2)-\frac{\mathcal{F}(x)+\mathcal{F}(y)}{2}.$$

Thus, using Equation (39) we obtain

$$\frac{\delta^{\frac{\beta}{k}}\Gamma_k(\beta+k)}{2(y-x)^{\frac{\beta}{k}}}\left[{}^{(k,\psi)}\mathcal{J}^{\beta,\delta}_{(\psi^{-1}(\mu_1+\mu_2-\frac{x+y}{2}))^+}(\mathcal{F}\circ\psi)(\psi^{-1}(\mu_1+\mu_2-x))\right.$$
$$\left.+{}^{(k,\psi)}\mathcal{J}^{\beta,\delta}_{(\psi^{-1}(\mu_1+\mu_2-\frac{x+y}{2}))^-}(\mathcal{F}\circ\psi)(\psi^{-1}(\mu_1+\mu_2-y))\right]$$
$$\leq \mathcal{F}(\mu_1)+\mathcal{F}(\mu_2)-\frac{\mathcal{F}(x)+\mathcal{F}(y)}{2}.$$

This completes the proof of the second inequality. □

4. Conclusions

Our results focus on a new generalization of Hermite–Hadamard–Mercer's inequalities for convex function by employing left-sided (k,ψ)-proportional fractional integral operators involving a continuous strictly increasing function. The obtained results are a generalization of Hermite–Hadamard–Mercer's inequality that are given through proportional fractional integral operators of integrable function with respect to another continuous and strictly increasing function [22].

Author Contributions: Conceptualization, H.D.D. and T.A.; writing—original draft preparation, H.D.D. and T.A.; writing—review and editing, H.D.D., J.B.M., T.A. and E.R.N.; supervision, E.R.N. and J.B.M. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Acknowledgments: Many thanks to the referees for their comments and suggestions. The first author thanks Addis Ababa University Department of Mathematics and International Science Program(ISP), Uppsala University(Sweden) for their support.

Conflicts of Interest: Authors have declared that they have no conflict of interest.

References

1. Mitrinović, D.; Lacković, I. Hermite and convexity. *Aequationes Math.* **1985**, *28*, 229–232. [CrossRef]
2. Beckenbach, E.F. Generalized convex functions. *Bull. Am. Math. Soc.* **1937**, *43*, 363–371. [CrossRef]
3. Sun, J.; Xi, B.Y.; Qi, F. Some new inequalities of the Hermite–Hadamard type for extended s-convex functions. *J. Comput. Anal. Appl.* **2019**, *26*, 985–996.
4. Nwaeze, E.R. Inequalities of the Hermite–Hadamard type for quasi-convex functions via the (k,s)-Riemann–Liouville fractional integrals. *Fract. Differ. Calc.* **2018**, *8*, 327–336. [CrossRef]
5. Khan, M.A.; Mohammad, N.; Nwaeze, E.R.; Chu, Y.M. Quantum Hermite–Hadamard inequality by means of a Green function. *Adv. Differ. Equ.* **2020**, *2020*, 1–20.

6. Almutairi, O.; Kılıçman, A. New generalized Hermite–Hadamard inequality and related integral inequalities involving Katugampola type fractional integrals. *Symmetry* **2020**, *12*, 568. [CrossRef]
7. Niculescu, C.; Persson, L.E. *Convex Functions and Their Applications*; Springer: Berlin/Heidelberg, Germany, 2006; Volume 23.
8. Mercer, A.M. A variant of Jensen's inequality. *J. Inequal. Pure Appl. Math.* **2003**, *4*, 73.
9. Abdeljawad, T.; Ali, M.A.; Mohammed, P.O.; Kashuri, A. On inequalities of Hermite–Hadamard–Mercer type involving Riemann–Liouville fractional integrals. *AIMS Math.* **2020**, *5*, 7316–7331. [CrossRef]
10. Butt, S.I.; Kashuri, A.; Umar, M.; Aslam, A.; Gao, W. Hermite–Jensen–Mercer type inequalities via Ψ-Riemann–Liouville k-fractional integrals. *AIMS Math.* **2020**, *5*, 5193–5220. [CrossRef]
11. Chu, H.H.; Rashid, S.; Hammouch, Z.; Chu, Y.M. New fractional estimates for Hermite–Hadamard–Mercer's type inequalities. *Alex. Eng. J.* **2020**, *59*, 3079–3089. [CrossRef]
12. Vivas-Cortez, M.; Ali, M.A.; Kashuri, A.; Budak, H. Generalizations of fractional Hermite–Hadamard–Mercer like inequalities for convex functions. *AIMS Math.* **2021**, *6*, 9397–9421. [CrossRef]
13. Faisal, S.; Adil Khan, M.; Khan, T.U.; Saeed, T.; Alshehri, A.M.; Nwaeze, E.R. New "Conticrete" Hermite–Hadamard–Jensen–Mercer Fractional Inequalities. *Symmetry* **2022**, *14*, 294. [CrossRef]
14. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; Elsevier: Amsterdam, The Netherlands, 2006; Volume 204.
15. Samko, S.G.; Kilbas, A.A.; Marichev, O.I. *Fractional Integrals and Derivatives*; Gordon and Breach Science Publishers: Yverdon Yverdon-les-Bains, Switzerland, 1993; Volume 1.
16. Jarad, F.; Abdeljawad, T.; Alzabut, J. Generalized fractional derivatives generated by a class of local proportional derivatives. *Eur. Phys. J. Spec. Top.* **2017**, *226*, 3457–3471. [CrossRef]
17. Kwun, Y.C.; Farid, G.; Nazeer, W.; Ullah, S.; Kang, S.M. Generalized Riemann–Liouville k-fractional integrals associated with Ostrowski type inequalities and error bounds of Hadamard inequalities. *IEEE Access* **2018**, *6*, 64946–64953. [CrossRef]
18. Diaz, R.; Pariguan, E. On hypergeometric functions and Pochhammer k-symbol. *arXiv* **2004**, arXiv:math/0405596.
19. Aljaaidi, T.A.; Pachpatte, D.B.; Abdo, M.S.; Botmart, T.; Ahmad, H.; Almalahi, M.A.; Redhwan, S.S. (k,ψ)-Proportional Fractional Integral Pólya–Szegö and Grüss-Type Inequalities. *Fractal Fract.* **2021**, *5*, 172. [CrossRef]
20. Öğulmüş, H.; Sarıkay, Z.M. Hermite–Hadamard–Mercer type inequalities for fractional integrals. *Filomat* **2021**, *35*, 2425–2436. [CrossRef]
21. Kian, M.; Moslehian, M. Refinements of the operator Jensen–Mercer inequality. *Electron. J. Linear Algebra* **2013**, *26*, 742–753. [CrossRef]
22. Aljaaidi, T.A.; Pachpatte, D.B. The Hermite–Hadamard–Mercer Type Inequalities via Generalized Proportional Fractional Integral Concerning Another Function. *Int. J. Math. Math. Sci.* **2022**, *2022*, 6716830 . [CrossRef]